Prelude

I hail a semi-group when I see one and I seem to see them everywhere!

With this statement Carl Einar Hille (1894-1980) opens the preface to the 1948 first edition of his monumental work "Functional Analysis and Semi-groups". While it is not unusual that mathematicians see their favourite subject "everywhere", there is a serious background behind Hille's creed, reaching much farther than expressing a mere subjective preference. I suggest the following interpretation, to be further specified in the sequel.

> **Semigroups are the proper codification of deterministic autonomous motion.**

Although the history of a mathematical analysis of motion goes back at least to Galileo Galilei (1564-1642) and Isaac Newton (1642-1727), the claimed connection between *semigroups* and *deterministic motion in time* is comparably new. We agree with Hille when he writes (see [Hi65]):

> *Like Monsieur Jourdain in* Le Bourgeois Gentilhomme*, who found to his great surprise that he had spoken prose all his life, mathematicians are becoming aware of the fact that they have used semi-groups extensively even if not always consciously. (...) The concept was formulated and named as recently as 1904, and it is such a primitive notion that one may well be in doubt concerning its value and possible implications.*

The implications *are* indeed farreaching and Hille substantiates this throughout his book. One major implication is, of course, the connection between deterministic motion and the semigroup structure. In fact, Jacques Salomon Hadamard (1865-1963) seems to be the first[1] to realise and express this connection. He discussed this in his famous 1923

[1]However, already in 1887 Giuseppe Peano (1858-1932) solved a system of first order ordinary differential equations by a matrix valued exponential function, thus giving a semigroup solution of a special case of an initial value problem [Pe97]. In 1910 his student Maria Gramegna († 1915) solved certain integrodifferential equations by semigroup methods [Gra10]. Thus Hadamard had precursors showing in concrete cases how a (wellposed, deterministic) evolution equation is solved by an exponential function, i.e., a semigroup. For a detailed discussion of the history of the exponential function see [EN00, Chap. VII.].

textbook “Lectures on Cauchys Problem” [Ha23] with reference to the great Dutch scholar Christiaan Huygens (1629-1695). His principle for the evolution process for luminous phenomena is analysed by Hadamard using the form of a syllogism, [Ha23, p. 53]. We cite here the major proposition.

> *(A) (major premise) The action of phenomena produced at the instant* 0 *on the state of matter at the later time* t_0 *takes place by the mediation of every intermediate instant* t'*, i.e. (...), in order to find out what takes place for* t_0*, we can deduce from the state at* 0 *the state at* t' *and, from the latter, the required state at* t_0*.*

The minor proposition states that initial conditions *localised* at the origin will cause an effect localised on a sphere around the origin. As the concluding proposition, Hadamard obtained the well known principle of light evolution by secondary waves. He emphasised that Huygens’ principle was often understood in quite different senses mixing up the above distinguished three propositions.

The major proposition (A), on which we will concentrate in the following, is called *priciple of scientific determinism* by Hadamard. He attributes it to the philosophical tradition as a *“law of thought”* and states (see [Ha23, p. 54])

> *(A) must therefore be considered as a truism, which does not mean that it cannot interest us; (...) the above proposition, in particular, corresponds to the fact that the integration of partial differential equations defines certain groups of functional operations (...)*

Solving a deterministic evolution equation thus yields a *group* of solution operators. We now explain this in more detail following [Ni00] and [Ni02].

We start from a concept of **‘motion’** designating, here and in what follows, any and all forms of temporal change. It is thus a much more general term than mere change of location. A mathematical framework for a description of motion can be chosen as follows[2]:

1. The object of inquiry is the **motion** of a **system** in **time**.

2. **Time** is represented by the (semi)group of real numbers $\mathbb{R}$ (or $\mathbb{R}_+$), respectively[3]. We thus use the structure of a one-dimensional, homogeneous, ordered continuum[4].

[2]We concentrate here on the case of (reversible) motion with continuous time and global existence. We also suppose a certain time regularity. However, more complicated behaviour — such as blow up — can basically be treated in a similar way. (Consider in that case, e.g., a bijection between the existence interval $(0,\tau)$ and $(-\infty,\infty)$).

[3]For technical reasons we will restrict the discussion to reversible motion, thus time is represented by $\mathbb{R}$.

[4]This identification is not so innocent as it might appear. While many criticisms of this definition could

3. The **system** under consideration is characterised by a set $\mathcal{Z}$ — the state space — of distinct states $z \in \mathcal{Z}$, whose temporal change is to be determined. The set of all possible states of the system is thus fixed from the outset.

   For example, the state space of a 'planetary system' could consist of the positions only or of positions and velocities (or momenta) of all planets; the state space of an 'eco-system' could be made up of the number of individuals belonging to each relevant species (or the respective positions as well); and the state space of a 'chemical system' could be composed of the concentrations of the relevant chemical substances[5].

4. The **motion** of the system is described by the temporal change of states, thus is represented by a function $\mathbb{R} \ni t \mapsto z(t) \in \mathcal{Z}$ mapping each instant $t \in \mathbb{R}$ to one and only one state $z(t) \in \mathcal{Z}$.

These characteristics describe the 'real motion' of a system as a mapping from the time space $\mathbb{R}$ into the state space $\mathcal{Z}$. Motion thus inherits basic properties of the presupposed structure of time.

Up to now we described only one motion of the system; no alternative route is taken into account. The observer outside the system can (at least theoretically) oversee this motion as a whole (by regarding the complete function $z(\cdot)$). The system itself has at no time another option to 'choose' than the prescribed one. This conceptual framework can thus be called *determinism with respect to real motion.* This setting, however, does not develop its full force until a perspective is adopted taking into account *all possible* motions. It can be called *determinism with respect to all possible motions.* This change of perspective is of major importance. Here the (human) observer steps finally out of the playground and describes the motion as if he could run the course of the world again and again[6].

5. For every instant $t_0 \in \mathcal{T}$ and every initial state $z_0 \in \mathcal{Z}$ there exists one and only one (thus necessarily determined) motion $z_{t_0,z_0} : \mathcal{T} \to \mathcal{Z}$ which at time t_0 yields the state z_0, i.e., $z_{t_0,z_0}(t_0) = z_0$.

By varying the initial time $t_0 \in \mathbb{R}$ and the intermediate times $t \in \mathbb{R}$ we obtain for the system under consideration a family of mappings $\Phi_{t,t_0} : \mathcal{Z} \to \mathcal{Z}$. Every function Φ_{t,t_0} maps an arbitrary state $z_0 \in \mathcal{Z}$ to the state $z_{t_0,z_0}(t)$, reached at time t by the unique motion

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be cited here, one from David Hume (1711 - 1776) will suffice: *"An infinite number of real parts of time, passing in succession, and exhausted one after the other, appears so evident a contradiction, that no man, one should think, whose judgement is not corrupted, instead of being improved, by the sciences, would ever be able to admit of it."* [Hu], p. 424.

[5] Also stochastic time evolutions in some original state space X may fit into this scheme by taking the space of probability densities $L^1(X)$ as new state space.

[6] This perspective together with the possibility of 'preparing' suitable initial states is also the condition of experimentation. It is therefore constitutive for modern natural science.

beginning at time t_0 just in state z_0. Formally, we can write

$$\Phi_{t,t_0}(z_0) := z_{t_0,z_0}(t).$$

Every state $z_1 = \Phi_{t_1,t_0}(z_0)$ can itself be regarded as an initial state. The motion determined by the pair (t_1, z_1) must coincide with the original motion. Otherwise there would be two different motions passing through (t_1, z_1) contradicting Condition 5. In terms of the mappings this is expressed by the equation

$$\Phi_{t,t_1}(z_1) = \Phi_{t,t_1}(\Phi_{t_1,t_0}(z_0)) = \Phi_{t,t_0}(z_0).$$

Since this holds for all $z_0 \in \mathcal{Z}$, we obtain the following fundamental equation

$$\Phi_{t,s} \circ \Phi_{s,r} = \Phi_{t,r} \tag{1}$$

for all $t, s, r \in \mathbb{R}$. Moreover,

$$\Phi_{t,t} = Id \tag{2}$$

holds with the identical mapping $Id : z \mapsto z$.

This construction can be called *deterministic* on the basis of the following characteristics: the a priori choice of a state space, the representation of time by the set of real numbers $\mathbb{R}$, and finally the necessary existence of a unique motion for every possible initial state (see Condition 5).

In mathematical terms, a deterministic motion is given by the state space $\mathcal{Z}$, time, and a family of mappings $\Phi_{t,s} : \mathcal{Z} \to \mathcal{Z}$ which fulfill the equations (1) and (2)[7].

For the sake of simplicity we now assume that the system is not subject to external influence in the course of time, so the motion is independent of any absolute point of time. Such systems are called autonomous[8]. Hence the momentary state of any motion depends solely on the initial state and the time *difference* between start and finish.

[7] The discussion of the implications of dissecting a motion into individual steps goes back at least to ARISTOTLE (384 - 322). In his lecture on nature, he sharply distinguishes between an *actual* interruption of movement (that of a *"mobile"*, i.e., a moving object along a line) and its mere *possibility: "(...) whereas any point between the extremities may be made to function dually in the sense explained (as beginning and as end, G.N.), it does not actually function unless the mobile actually divides the line by stopping and beginning to move again. Else there were one movement, not two, for it is just this that erects the 'point between' into a beginning and an end (...)"* [A], p. 373. In case a continual motion occurs, then there is no justification for saying the object is in the middle position (during a given period of time): *"But if anyone should say that it (A, G.N.) has 'arrived' at every potential division in succession and 'departed' from it, he will have to assert that as it moved it was continually coming to a stand. For it cannot 'have arrived' at a point (B, G.N.) (which implies that it is there) and 'have departed' from it (which implies that it is not there) at the same point in time. So there are two points of time concerned, with a period of time between them; and consequently A will be at rest at B (...)"* [A], p. 375. From this quite consistent perspective, the deduction of a relation as given in (1) certainly seems problematic.

[8] Every system whatsoever can be embedded into a larger autonomous system by integrating the changing environment into the system until external change is eliminated. For a corresponding mathematical procedure for associating an autonomous system, see [Ni96].

In this context it is remarkable that even HADAMARD did not distinguish carefully between autonomous and non-autonomous motion[9]. The identity $\Phi_{t,s} = \Phi_{t+\tau,s+\tau}$ is thus valid for any $\tau \in \mathfrak{T}$, and a unique mapping

$$T_t : \mathcal{Z} \to \mathcal{Z}, \quad T_t := \Phi_{t,0} = \Phi_{t+\tau,\tau}$$

can be defined. Then T_t maps every initial state z_0 to the final state $z_1 = T_t(z_0)$, depending only on the elapsed time difference t. The immediate consequences of equations (1) and (2) for the mappings T are

$$T_s T_t = T_{t+s}, \quad T_0 = Id. \tag{3}$$

A family of mappings which fulfills equation (3) is called either a **one-parameter group** (for $t \in \mathbb{R}$) or a **one-parameter semigroup** (for $t \in \mathbb{R}_+$). *The structure of a one-parameter semigroup is therefore a mathematical model of autonomous, deterministic motion.*

Instead of going into the details of the long lasting philosophical debate on the concept of determinism (parts of it may be found in [Ni00, Ni02]), I will discuss here only one detail. The above connection between *semigroup* and *scientific determinism* is so evident that it seems strange that it took approx. 300 years from the beginning of a mathematical analysis of motion (with GALILEI and NEWTON) to its precise statement by HADAMARD.

As a first attempt for an explanation we remark that the explicit formulation of an *abstract* concept is in general the final point after a long time of implicitely using it in *concrete* situations. In fact, the notion of a *group* or *semigroup* is an invention of the 20^{th} century. However, by this 'explanation' we only reformulated the question.

There is, however, a second aspect, which we will consider more precisely, decisive for the developement of the theory of dynamical systems. The concept of a *state space* is relatively new in the history of dynamical systems. More or less up to the rise of quantum mechanics (and, less important, statistical mechanics) it was evident that the 'real' space containing any material system is a three dimensional Euclidian space[10]. Thus also HUYGENS' principle was taken only for the special case of light evolution in $\mathbb{R}^3$. Only the distinction into three different propositions by HADAMARD allows to see its general importance.

This prevalent concept of space[11] is supported from the philosophical point of view, e.g., by IMMANUEL KANT (1724-1804)[12]. In his *Kritik der reinen Vernunft*, e.g., we find the

[9]This distinction is given in the mathematical physics textbook of RICHARD COURANT (1888-1972) and DAVID HILBERT (1862-1943). It is interesting that an abstract treatment of evolution families solving a deterministic evolution equation similarly to our presentation below does not appear in the first and second German (vol. I 1924/ vol. II 1937, 1968) or the first English edition (1953). It is only the second English edition (1962) presenting this abstract result.

[10]Also in textbooks of philosophy of science the emphasis is often on $\mathbb{R}^3$ and related concepts, compare, e.g., [Ea86]

[11]For a comprehensive treatment of the history of physical space concepts see, e.g., [Ja93].

[12]More precisely: By an easy 'physicist's interpretation of his philosophy.

statement:

> *Der Raum ist kein empirischer Begriff, der von äußeren Erfahrungen abgezogen worden. (...) Der Raum ist eine notwendige Vorstellung, a priori, die allen äußeren Anschauungen zum Grunde liegt*[13]. KrV B39

For KANT all theorems of the Euclidian geometry will thus hold a priori:

> *So werden auch alle geometrischen Grundsätze, z.E. daß in einem Triangel zwei Seiten zusammen größer sind als die dritte, niemals aus allgemeinen Begriffen (...) sondern aus der Anschauung und zwar a priori mit apodiktischer Gewißheit abgeleitet*[14]. KrV A25.

Also the dimension of space is given *a priori* and equal to three. This concept of space is also basic for his definition of motion. In *Metaphysische Anfangsgründe der Naturwissenschaft* KANT writes:

> *Bewegung eines Dinges ist die Veränderung der* äußeren *Verhältnisse desselben zu einem gegebenen Raum*[15]. MA A5

Any seemingly *inner* motion of an object — KANT takes as an example the fermentation of a barrel of beer — must be reduced to motion in the above sense. We obtain thus a framework for natural science in which whatsover kind of motion has to be formulated, in the end, in $\mathbb{R}^3$. This goes so far that the biologist, physiologist and philosopher EMIL DU BOIS–REYMOND (1818 - 1896), cited here as a witness for classical natural science, can state (see [Du12a]):

> *Kant's Behauptung in der Vorrede zu den Metaphysischen Anfangsgründen der Naturwissenschaft, 'daß in jeder besonderen Naturlehre nur so viel eigentliche Wissenschaft angetroffen werden könne, als darin Mathematik anzutreffen sei' — ist also vielmehr noch dahin zu verschärfen, daß für Mathematik Mechanik der Atome [in $\mathbb{R}^3$, G.N.] zu setzen ist*[16].

[13] Space is not an empirical concept which has been derived from outer experiences. (...) Space is a necessary a priori representation, which underlies all outer intuitions (transl. N. Kemp Smith).

[14] For kindred reasons, geometrical propositions, that, for instance, in a triangle two sides together are greater than the third, can never be derived from the general concepts of line and triangle, but only from intuition, and this indeed a priori, with apodeictic certainty (transl. N. Kemp Smith).

[15] Motion of an object is change of the outer relations of it with respect to a given space (transl. G.N.).

[16] Kant's claim in the preface to the metaphysical foundations of natural sciences, 'that in every special natural doctrine there is only as much real science as there is matheamtics' must be formulated stricter insofar, that one must replace mathematics by mechanics of atoms [in $\mathbb{R}^3$, G.N.] (transl. G.N.).

The state spaces used in analytical mechanics, e.g., $\mathbb{R}^{6N}$ for an N-particle system, are considered only for mathematical calculations without any further interpretation. With the rise of quantum mechanics[17] the question for the correct state space gains much more interest. For instance, state and observable have to be carefully distinguished. Moreover, the non-Euclidean geometries further foster the impression that $\mathbb{R}^3$ is by no means the only possible state space.

However, the choice of the state space is of major importance for the properties of a dynamical system. The easy example of a Newtonian $N-$particle system in $\mathbb{R}^{3N}$ (non-deterministic) or $\mathbb{R}^{6N}$ (deterministic) illustrates this fact. The philosopher Ernst Cassirer (1874-1945) states in his great essay "Determinismus und Indeterminismus in der Modernen Physik", [Ca32]:

> *Die Antwort auf das Kausalproblem, die eine naturwissenschaftliche Erkenntnislehre uns gibt, steht niemals für sich allein, sondern sie beruht stets auf einer bestimmten Annahme über den naturwissenschaftlichen Objektbegriff. Beide Momente greifen unmittelbar ineinander ein und bedingen sich wechselseitig.*[18]

By a sloppy identification of state space and concept of object we might say: The state space of a system is not a priori fixed, making *afterwards* the systems evolution deterministic or not. Rather the contrary holds: We first require that a system should evolve deterministically, thus the momentary state determines the whole motion; then we try to find a state space guaranteeing this property. Essentially the claim of determinism predetermines the possible choices for a state space.

A divorce from the prejudice that 'real' space is a Euclidian $\mathbb{R}^3$ seems crucial for a chance to realising the fundamental structures of a mathematical description of motion.

As we saw above, every deterministic (autonomous) evolution leads to a semigroup after the appropriate choice of a state space. It is, however, not only the structural viewpoint which leads to a semigroup approach. Adding more structure for the state space, e.g., choosing an (infinite dimensional) Banach space as the state space, there are powerful analytical means at hand to studying the *qualitative* behaviour of the system. The theory of strongly continuous semigroups on Banach spaces offers a broad range of these techniques for the analysis of qualitative and asymptotic behaviour, and this approach has gained great popularity in recent years (see [EN00, Chap. V]).

We now come back to the opening phrase in the preface of Hille's book which unfortunately continues as follows:

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[17] It is thus not surprising that Stone's theorem, formulated 1930, gives the solution of the deterministic evolution equation of a quantum system by a *group* of operators, see [St32].

[18] *The answer that an epistemology of science gives to the problem of causality [i.e. determinism (G.N.)] never stands alone but always depends on a certain assumption as to the nature of the object in science. These two are intimately connected and mutually determine each other (transl. O. T. Benfey).*

> *Friends have observed, however, that there are mathematical objects which are not semi-groups.*

In the spirit of our previous discussion we suggest the following interpretation of this statement

> There are many evolution equations arising from concrete and important applications where you cannot see a semigroup of solution operators.

We will now give some rather abstract examples for this phenomenon.

Example 0.0.1 (Nonautonomous equations). *Consider some Banach space X, a family of linear operators $(A(t), D(A(t)))_{t\in\mathbb{R}}$, and the nonautonomous Cauchy problem*

$$\text{(NCP)} \qquad \begin{cases} \dot{u}(t) &= A(t)u(t), \qquad t \geq s, \\ u(s) &= x. \end{cases}$$

The solutions are given by an evolution family, *but not by a semigroup (cf. our above discussion).*

Example 0.0.2 (Second order equations). *Consider some Banach space X and the second order abstract Cauchy problem*

$$(\mathrm{ACP}_2) \qquad \begin{cases} \ddot{u}(t) &= B\dot{u}(t) + Au(t), \qquad t \geq 0, \\ u(0) &= x \in X, \\ \dot{u}(0) &= y \in X, \end{cases}$$

with linear operators $(A, D(A))$ and $(B, D(B))$ on X. Again, on the state space X there is no semigroup of solutions (see [EN00, Sect. VI.3]).

Example 0.0.3 (Delay equations). *Consider some Banach space ∂X and an abstract delay equation*

$$\text{(DE)} \qquad \begin{cases} \dot{x}(t) &= Bx(t) + \Phi x_t, \quad t \geq 0, \\ x_0 &= g \in L^p([-1,0], \partial X), \\ x(0) &= y \in \partial X \end{cases}$$

on ∂X, where $(B, D(B))$ is a generator on the Banach space ∂X, $\Phi : L^p([-1,0], \partial X) \to \partial X$ is the **delay operator** *and, as usual, the* **history function** *is defined by $x_t(\tau) := x(t+\tau)$ (see [HVL93, BP03]). Again, there is no solution semigroup on ∂X.*

Example 0.0.4 (Dynamic boundary value problems). *Consider any bounded region $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$ and a diffusion process on Ω as well as on the boundary*

$\partial\Omega$. This leads to the dynamic boundary value problem

$$(\mathrm{BP}) \quad \begin{cases} \dot f(t,\xi) &=& \Delta_\Omega f(t,\xi), \quad \xi \in \Omega,\ t \geq 0, \\ x(t,\eta) &:=& f(t,\eta), \quad \eta \in \partial\Omega,\ t \geq 0, \\ \dot x(t,\eta) &=& \Delta_{\partial\Omega} x(t,\eta), \quad \eta \in \partial\Omega,\ t \geq 0, \\ f(0,\cdot) &=& g \in L^2(\Omega), \\ x(0,\cdot) &=& y \in L^2(\partial\Omega). \end{cases}$$

This type of evolution equation occurs in many situations and has been studied intensively with various techniques and goals. As a main source for motivation and numerous applications we refer to [LT00] where the emphasis is on boundary control problems. Our example is taken from [CENN01]. Of course, without the dynamic equation on the boundary and with suitable (e.g., Dirichlet) boundary condition the Laplacian on Ω becomes a generator and we obtain a semigroup solution on $L^2(\Omega)$. However, the dynamic process on the boundary destroys this property.

By the leading principle emphasised in the preceding section — the state space determines determinism — it is not surprising that we have to change the state space in order to regain a deterministic motion for the above examples. In fact, e.g., the state space $C_0(\mathbb{R}, X)$ allows a semigroup treatment of the nonautonomous equations of Example 0.0.1 (see [EN00, Sect. VI.9]), the state space $X \times [D(B)]$ enables a semigroup treatment of the second order equations in Example 0.0.2 (see [EN00, Sect. VI.3]), and the state space for Example 0.0.3 may be choosen as $\mathcal{X} := L^p([-1,0], \partial X) \times \partial X$ (see [BP03]). In Example 0.0.4 we may choose a product space $\mathcal{X} := L^2(\Omega) \times L^2(\partial\Omega)$ as new state spaces in order to obtain a solution semigroup (see this thesis).

Thus the following chapters are devoted to substantiate, e.g., in the situation of Example 0.0.3 and 0.0.4 the following modified interpretation of Hille's opening phrases:

If you cannot see the semigroup behind an evolution equation, take a closer look and you will.