NIELSEN COINCIDENCE THEORY IN ARBITRARY CODIMENSIONS

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ABSTRACT. Given two maps $f_1, f_2 : M^m \longrightarrow N^n$ between manifolds of the indicated arbitrary dimensions, when can they be deformed away from one another? More generally: what is the minimum number $MCC(f_1, f_2)$ of *pathcomponents* of the coincidence space of maps f'_1, f'_2 where f'_i is homotopic to f_i, $i = 1, 2$? Approaching this question via normal bordism theory we define a lower bound $N(f_1, f_2)$ which generalizes the Nielsen number studied in classical fixed point and coincidence theory (where $m = n$). In at least three settings $N(f_1, f_2)$ turns out to coincide with $MCC(f_1, f_2)$: (i) when $m < 2n - 2$; (ii) when N is the unit circle; and (iii) when M and N are spheres and a certain injectivity condition involving James-Hopf invariants is satisfied. We also exhibit situations where $N(f_1, f_2)$ vanishes, but $MCC(f_1, f_2)$ is strictly positive.

1. Introduction and statement of results

Throughout this paper $f_1, f_2 : M \longrightarrow N$ denote two (continuous) maps between given smooth connected manifolds M and N without boundary, of strictly positive dimensions m and n, resp., M being compact. (All manifolds are assumed Hausdorff with a countable basis).

Definition 1.1. We say the pair (f_1, f_2) is *loose* if there exist (continuous) maps f'_i homotopic to f_i, $i = 1, 2$, without coincidences (i.e. $f'_1(x) \neq f'_2(x)$ for all $x \in M$) or, equivalently, if the **m***inimum number of* **c***oincidence* **c***omponents*

$$MCC(f_1, f_2) \ := \ \min\{\#\pi_0(C(f'_1, f'_2)) | f'_1 \sim f_1, f'_2 \sim f_2\}$$

vanishes. Here $\#\pi_0(C(f'_1, f'_2))$ denotes the number of *pathcomponents* of the coincidence subspace

$$C(f'_1, f'_2) \ = \ \{x \in M | f'_1(x) = f'_2(x)\}$$

of M.

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Question. When is (f_1, f_2) loose? In other words: when can the maps f_1 and f_2 be deformed away from one another? More generally, how big is $MCC(f_1, f_2)$?

In many interesting cases an answer can be given with the help of obstructions $\widetilde{\omega}(f_1, f_2)$ and $N(f_1, f_2)$ which we will now describe.

An analysis of the coincidence behaviour (of a suitable approximation) of $(f_1, f_2) : M \longrightarrow N \times N$ has led, in [K 3], to a normal bordism class

$$\omega(f_1, f_2) = [C, g, \overline{g}] \in \Omega_{m-n}(M; \varphi) \tag{1.2}$$

where C is essentially the coincidence locus and $\overline{g}$ is a vector bundle isomorphism which describes the stable normal bundle of C in terms of a pullback of the virtual coefficient bundle

$$\varphi := f_1^*(TN) - TM \tag{1.3}$$

over M. (Thus e.g. if M and N are stably parallelized, then $\omega(f_1, f_2)$ lies in the framed bordism group $\Omega^{fr}_{m-n}(M)$).

In the present paper we will study a considerably sharper invariant inspired by the work of Hatcher and Quinn [HQ]. Consider the commuting diagram

$$\begin{array}{ccc} & \overset{\widetilde{g}}{\nearrow} & E(f_1, f_2) \\ & & \big\downarrow \text{pr} \\ C & \xrightarrow[g = \text{incl}]{} & M \end{array} \tag{1.4}$$

where $E(f_1, f_2)$ *) consists of all pairs (x, θ) such that $x \in M$ and $\theta : [0,1] \longrightarrow N$ is a continuous path with $\theta(0) = f_1(x)$ and $\theta(1) = f_2(x)$; pr denotes the obvious projection and $\widetilde{g}$ is the natural lifting which adds the constant path at $f_1(x) = f_2(x)$ to $g(x) = x \in C$. Now, in general the topological space $E(f_1, f_2)$ is not pathconnected; in fact, its pathcomponents correspond bijectively to Reidemeister classes, i.e. to the classes in $\pi_1(N)$ with respect to a certain relation depending on f_1 and f_2 (see prop. 2.1 below). Thus the normal bordism class

$$\widetilde{\omega}(f_1, f_2) := [C, \widetilde{g}, \overline{g}] \in \Omega_{m-n}(E(f_1, f_2); \widetilde{\varphi}) = \bigoplus_{A \in \pi_0(E(f_1, f_2))} \Omega_{m-n}(A; \widetilde{\varphi}|A) \tag{1.5}$$

(where

$$\widetilde{\varphi} := pr^*(\varphi) \;) \tag{1.6}$$

can be decomposed into its contributions

$$\widetilde{\omega}_A(f_1, f_2) = [\; C_A := \widetilde{g}^{-1}(A),\; \widetilde{g}|C_A,\; \overline{g}| \;] \in \Omega_{m-n}(A; \widetilde{\varphi}|A) \tag{1.7}$$

to the various pathcomponents A.

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*) this differs slightly from the notation of Hatcher and Quinn.

Definition 1.8. A pathcomponent A of $E(f_1, f_2)$ is called *essential* if the corresponding direct summand of $\widetilde{\omega}(f_1, f_2)$ is nontrivial.

The *Nielsen coincidence number* $N(f_1, f_2)$ is the number of essential pathcomponents $A \in \pi_0(E(f_1, f_2))$.

Since we assume M to be compact, $N(f_1, f_2)$ is finite.

The Nielsen number is a simple (but crude) measure for the non-triviality of the much more delicate invariant $\widetilde{\omega}(f_1, f_2)$ which lies, a priori, in a group varying with f_1 and f_2. However, clearly $\widetilde{\omega}(f_1, f_2)$ vanishes if and only if $N(f_1, f_2)$ does.

Theorem 1.9. *Given any pair of maps $f_1, f_2 : M \to N$ we have:*

(i) $N(f_1, f_2)$ *depends only on the homotopy classes of* f_1 *and* f_2*;*
(ii) $N(f_1, f_2) = N(f_2, f_1)$*;*
(iii) $N(f_1, f_2) \leq MCC(f_1, f_2) < \infty$ (*cf.* 1.1)*;*
if $n \neq 2$, *then also* $MCC(f_1, f_2) \leq \#\pi_0(E(f_1, f_2))$ (*i.e.* $MCC(f_1, f_2)$ *is bounded by the cardinality of the Reidemeister set of* (f_1, f_2), *cf.* 2.1).

In particular, it follows that $N(f_1, f_2)$ must vanish if (f_1, f_2) is loose (cf. 1.1). This leads us to ask the more specific

Question. Is the Nielsen number $N(f_1, f_2)$ the only looseness obstruction for (f_1, f_2)? More generally: is $N(f_1, f_2)$ equal to $MCC(f_1, f_2)$?

As we will see below (cf. 1.10, 1.13, and 1.15) the answer is positive in at least three interesting settings, but not always (cf. 1.17).

Theorem 1.10. *Assume* $m < 2n - 2$.
Then for all maps $f_1, f_2 : M^m \to N^n$ *we have* $MCC(f_1, f_2) = N(f_1, f_2)$.
In particular, (f_1, f_2) *is loose if and only if* $N(f_1, f_2) = 0$.

The proof (given in section 4 below) emphasizes the role of the paths θ which occur in $E(f_1, f_2)$ and in any partial nulbordism of $(C, \widetilde{g}, \overline{g})$ (cf. 1.4 and 1.5): they are a crucial ingredient in the construction of homotopies which eliminate inessential Nielsen coincidence classes.

Example I: $m = n$ (the classical case). Here $\widetilde{\omega}(f_1, f_2)$ lies in

$$\Omega_0(E(f_1, f_2); \widetilde{\varphi}) = \bigoplus_{\substack{A \in \pi_0(E(f_1,f_2)) \\ \text{with } \widetilde{\varphi}|A \\ \text{oriented}}} \mathbb{Z} \quad \oplus \bigoplus_{\substack{A \in \pi_0(E(f_1,f_2)) \\ \text{with } \widetilde{\varphi}|A \\ \text{non orientable}}} \mathbb{Z}_2$$

and counts the (generically transverse) coincidence points with or without signs. (The normal bordism approach makes it clear from the outset which of these two possibilities apply; in proposition 5.2 below we will give an explicit natural criterion in terms of fundamental groups). At least for orientable M and N the resulting contribution to a pathcomponent A is just the index of the corresponding Reidemeister (or Nielsen) class (cf. e.g. [BGZ], 3.5 – 3.6) and our definition of the Nielsen number agrees with the classical one. Moreover, a modified version of theorem 1.10 (cf. also proposition 4.7 below and [Br]) yields the classical "Wecken theorem" for closed smooth manifolds of dimension at least 3 (cf. [B], p. 12).

The central idea of Nielsen fixed point theory – to interpret the Lefschetz number (which corresponds to the ω-invariant, cf. 1.2) as the sum of indices of the various

Nielsen classes – is expressed here by the 0-dimensional normal bordism group of a single space. For $m - n > 0$ and arbitrary (f_1, f_2) our approach seems to be even better suited to capture relevant geometric aspects: $\widetilde{\omega}_A(f_1, f_2)$ (cf. 1.7) will, in general, reflect much more than the oriented or unoriented bordism class of the underlying partial coincidence manifold C_A or some derived (co)homology classes; often the full combined information contained in C_A, the map $\widetilde{g}|C_A$ and, in particular, the "twisted framing" $\overline{g}|$ turns out to be decisive (as illustrated e.g. by the examples in [K 3]).

Example II: $f_1 = f_2 =: f$ (selfcoincidence). Here the projection pr (cf. 1.4) allows a global section involving constant paths. Therefore $\widetilde{\omega}(f, f)$ (cf. 1.5) is precisely as strong as the obstruction $\omega(f) := \omega(f, f)$ (cf. 1.2) which was discussed in detail in [K 3]. In particular, $N(f, f)$ equals 0 or 1 according as $\omega(f)$ vanishes or not (since M is assumed to be connected). Also clearly $MCC(f, f) = 1$ except when (f, f) is loose (and hence $MCC(f, f) = 0$). □

An important special case of the $\widetilde{\omega}$–invariant is the *refined (normal bordism) degree*

$$(1.11) \qquad \widetilde{\deg}(f) := \widetilde{\omega}(f, *) \quad \in \quad \Omega_{m-n}(E(f, *); pr^*(f^*(TN) - TM))$$

of a map $f : M \longrightarrow N$. Here $*$ denotes a constant map; its choice is not very significant since any path in N from $*$ to $*'$, say, induces a fiber homotopy equivalence $E(f, *) \cong E(f, *')$ and an isomorphism of the corresponding normal bordism groups which takes $\widetilde{\omega}(f, *)$ to $\widetilde{\omega}(f, *')$. In particular, $\widetilde{\omega}(f, *)$ is compatible with the transitive action of $\pi_1(N, *)$ on $\pi_0(E(f, *))$. Therefore the pathcomponents of $E(f, *)$ are either all essential (i.e. $N(f, *) = \#\pi_0(E(f, *)) = \#(\pi_1(N)/f_*(\pi_1(M)))$; in this case also $MCC(f, *) = N(f, *)$ if $n \neq 2$) or all inessential (i.e. $N(f, *) = 0$; this holds e.g. if $\pi_1(N)/f_*(\pi_1(M))$ is infinite).

The invariant $\widetilde{\deg}(f)$ sharpens the (normal bordism) degree

$$(1.12) \qquad \deg(f) := \omega(f, *) \quad \in \quad \Omega_{m-n}(M; f^*(TN) - TM)$$

which was discussed extensively in [K 3].

Considerations of degrees (or "roots") are often very useful in the general coincidence setting, e.g. when n is low (and when therefore theorem 1.10 offers little insight).

Example III: maps into the unit circle.

Theorem 1.13. *Assume $N = S^1$. Then the Nielsen number is characterized by the identity*

$$(f_{1*} - f_{2*})(H_1(M; \mathbb{Z})) = N(f_1, f_2) \cdot H_1(S^1; \mathbb{Z}) \quad ,$$

and we have

$$MCC(f_1, f_2) \;=\; N(f_1, f_2)$$

(*cf. definition* 1.1).

Moreover, the following conditions are equivalent (without any dimension restriction)

(i) f_1 *and* f_2 *are homotopic;*

(ii) (f_1, f_2) *is loose* (*cf. definition* 1.1)*;*
(iii) *the Nielsen number* $N(f_1, f_2)$ *vanishes;*
(iv) $0 = \widetilde{\omega}(f_1, f_2) \in \Omega_{m-1}(E(f_1, f_2); -pr_*(TM))$*;*
(v) $0 = \omega(f_1, f_2) \in \Omega_{m-1}(M; -TM)$*; and*
(vi) $0 = \mu(\omega(f_1, f_2)) \in H_{m-1}(M; \widetilde{\mathbb{Z}_M}) \underset{PD}{\cong} H^1(M; \mathbb{Z})$

(where μ *and* PD *denote the obvious Hurewicz and Poincaré duality homomorphisms).*

In this very special situation there is no need to do any calculations of normal bordism invariants: since S^1 is a Lie group and an Eilenberg–MacLane space, the Nielsen number $N(f_1, f_2)$ can be computed easily via degree theory (cf. 1.11 – 1.12) and singular (co–)homology (and actually may assume all nonnegative integer values, for suitable choices of M, f_1 and f_2).

The detailed discussion of this example in § 6 will also include a simple and explicit description of the fiber homotopy type of $E(f_1, f_2)$ (which turns out to vary considerably, depending on $f_{1*} - f_{2*} : H_1(M; \mathbb{Z}) \to H_1(S^1; \mathbb{Z})$). □

Example IV: maps from spheres to spheres. Let $M = S^m$, $N = S^n$ with $n \geq 2$, and pick $y_0 \in S^n$. Then S^n is 1-connected and hence the Nielsen number $N(f_1, f_2)$ equals 0 or 1 according as $\widetilde{\omega}(f_1, f_2)$ vanishes or not. Thus we really have to study the $\widetilde{\omega}$–invariant in detail here. After a canonical identification all $\widetilde{\omega}$–invariants and the refined degree $\widetilde{\deg}(f)$ (compare 1.11) lie in the same group, namely the framed bordism group $\Omega^{fr}_{m-n}(\Lambda(S^n, y_0))$ of the loop space of (S^n, y_0), which is independent of $f_1, f_2, f \dots$.

Theorem 1.14. *Given dimensions* m *and* $n \geq 2$*, the refined* (*normal bordism*) *degree determines (and is determined by) a homomorphism which fits into the commuting diagram*

$$\begin{array}{ccc} [S^m, S^n] & \xrightarrow{\widetilde{\deg}} & \Omega^{fr}_{m-n}(\Lambda(S^n, y_0)) \\ \cong \uparrow & & \cong \downarrow h = \bigoplus_{k \geq 1} h_k \\ \pi_m(S^n) & \xrightarrow{\Gamma := \oplus E^\infty \circ \gamma_k} & \bigoplus_{k \geq 1} \pi^S_{m-1-k(n-1)} \end{array} .$$

Here $E^\infty \circ \gamma_k$ *denotes the stabilized* k^{th} *James-Hopf invariant homomorphism into the indicated stable homotopy groups of spheres,* $k = 1, 2, \dots;$ h_k *is a geometric homomorphism which measures* $(k-1)$*–fold selfintersections of suitable immersions.*

Given maps f_1, f_2, f *from* S^m *to* S^n*,* h *yields a decomposition of* $\widetilde{\omega}(f_1, f_2)$ *and* $\widetilde{\deg}(f)$ *into a sequence of components*

$$\widetilde{\omega}_k(f_1, f_2) \; := \; h_k(\widetilde{\omega}(f_1, f_2)), \quad \widetilde{\deg}_k(f) := h_k(\widetilde{\deg}(f)) \quad \in \quad \pi^S_{m-1-k(n-1)} ,$$

$k = 1, 2, \dots,$ *starting with the (Pontryagin-Thom isomorphism evaluated on the nonrefined) invariants* $\omega(f_1, f_2)$ *and* $\deg(f)$ *= Freudenthal suspension* $E^\infty(f)$ *of* f*, resp. We have for all* $k \geq 1$

$$\widetilde{\omega}_k(f_1, f_2) \; = \; \widetilde{\deg}_k(f_1) - (-1)^{k(n-1)} \widetilde{\deg}_k(f_2) \; ;$$

moreover, if $n \not\equiv k \equiv 0(2)$ *or if* $n \equiv 0(2)$ *and* $k \equiv 3$ *or* $4(4)$*, then* $2\widetilde{\omega}_k(f_1, f_2) = 0$.

We have always

$$MCC(f_1, f_2) = \begin{cases} 0 & \text{if } f_1 \sim a \circ f_2 \ , \\ 1 & \text{otherwise} \ ; \end{cases}$$

here a *denotes the antipodal map on* S^n*.*

Corollary 1.15. *Assume that the total stabilized James-Hopf homomorphism* Γ *(cf.* 1.14*) is injective on* $\pi_m(S^n)$*.*

Then for any two maps $f_1, f_2 : S^m \longrightarrow S^n$ *we have* $MCC(f_1, f_2) = N(f_1, f_2)$*. In particular the following conditions are equivalent:*

(i) (f_1, f_2) *is loose ;*
(ii) $N(f_1, f_2) = 0$ *; and*
(iii) f_1 *is homotopic to* $a \circ f_2$ *(where* $a : S^n \longrightarrow S^n$ *denotes the antipodal map).*

By Freudenthal's theorem, the assumption of this corollary is satisfied in the stable range $m < 2n - 1$ (where Γ consists only of the stable suspension E^∞) and hence is less restrictive than the dimension condition in theorem 1.10. With the added help of (higher) James-Hopf invariants we can answer our original question also in many nonstable dimension settings.

Corollary 1.16. *If* $m - n \leq 3$ *and* $n \geq 1$*, then* Γ *is injective on* $\pi_m(S^n)$ *and therefore the conclusion of corollary* 1.15 *holds.*

However, the diagram in theorem 1.14 can also lead to opposite results.

Corollary 1.17. *If* $n \neq 1, 3, 7$ *is odd and* $m = 2n - 1$*, or if e.g.* $(m, n) = (8, 4), (9, 4), (9, 3), (10, 4), (16, 8), (17, 8), (10 + n, n)$ *for* $3 \leq n \leq 11$*, or* $(24, 6)$*, then there exists a map* $f : S^m \longrightarrow S^n$ *such that the pair* $(f$*, constant map) is not loose although its Nielsen number vanishes.*

Remark 1.18. In a future paper we will also study the minimum number

$$MC(f_1, f_2) \ := \ \min\{\#C(f_1', f_2') | f_1' \sim f_1, f_2' \sim f_2\}$$

of coincidence *points* which plays a central role in classical fixed point and coincidence theory (where $m = n$; cf. e.g. [B] and [BGZ]). However, when $m > n$ then $MC(f_1, f_2)$ is very often infinite, and it seems more natural and illuminating to investigate $MCC(f_1, f_2)$ (cf. 1.1).

Remark 1.19. In classical fixed point theory it took 57 years to disprove the so-called Nielsen conjecture (which says that the minimum number MC, cf. 1.18, agrees with the Nielsen number; cf. [B],p. 12–14, and [Br]). The counterexamples to its higher-codimensional analogue provided by corollary 1.17 could be viewed as an indication that our generalized Nielsen number is too weak. In order to strengthen it, we may keep track e.g. of the fact that the map g (cf. 1.4) is an *embedding*, and of the *nonstable* vector bundle isomorphism $\nu(C(f_1, f_2), M) \cong f_1^*(TN)|$ (compare 4.3). This procedure can make our invariants considerably sharper but also harder to handle (cf. e.g. [K 6]).

Conventions. A framing of a smooth immersion or embedding is a (nonstable) trivialization of its normal bundle; a framing of a smooth manifold is a *stable* trivialization of its tangent bundle.

2. The space $E(f_1, f_2)$

Let $P(N)$ denote the space of all continuous paths $\theta : [0,1] \to N$, endowed with the compact-open topology. Then the projection $pr : E(f_1, f_2) \to M$ (cf. 1.4) is just the pullback of the starting point/end point fibration $P(N) \to N \times N$ by the map $(f_1, f_2) : M \to N \times N$.

In the next result we assume that there is a coincidence point $x_0 \in M$ and we put $y_0 := f_1(x_0) = f_2(x_0) \in N$ and $\theta_0 :=$ constant path at y_0. (If no such point exists, (f_1, f_2) is loose and our original question is answered). We will identify the fiber of pr at x_0 with the *loop space* $\Lambda(N, y_0)$ of paths in N starting and ending at y_0. Denote the fiber inclusion by incl.

Proposition 2.1. *The sequence of group homomorphisms*

$$\cdots \to \pi_{k+1}(M, x_0) \xrightarrow{f_{1*} - f_{2*}} \pi_{k+1}(N, y_0) \xrightarrow{\text{incl}_*} \pi_k(E(f_1, f_2), (x_0, \theta_0)) \xrightarrow{pr_*} \pi_k(M, x_0) \to \cdots$$
$$\cdots \longrightarrow \pi_1(M, x_0)$$

is exact. Moreover the map

$$\text{incl}_* \ : \ \pi_1(N, y_0) = \pi_0(\Lambda(N, y_0)) \longrightarrow \pi_0(E(f_1, f_2))$$

induces a bijection from the so called Reidemeister set

$$R(f_1, f_2, x_0) := \pi_1(N, y_0) \ / \ \text{Reidemeister equivalence}$$

onto the set of pathcomponents of $E(f_1, f_2)$. (We call $[\theta], [\theta'] \in \pi_1(N, y_0)$ Reidemeister equivalent *if $[\theta'] = f_{1*}(\mu)^{-1} \cdot [\theta] \cdot f_{2*}(\mu)$ for some $\mu \in \pi_1(M, x_0)$).*

Given a loop $\theta \in \Lambda(N, y_0)$ and the corresponding pathcomponent A_θ of (x_0, θ) in $E(f_1, f_2)$, we have

$$pr_*(\pi_1(A_\theta; (x_0, \theta))) = \{\mu \in \pi_1(M, x_0) | f_{2*}(\mu) = [\theta]^{-1} \cdot f_{1*}(\mu) \cdot [\theta]\} \ .$$

Proof. We are dealing here basically with the exact homotopy sequence of the Hurewicz fibration pr. The special form of the boundary homomorphism and the surjectivity of incl_* follow from the obvious

Fact 2.2. Given $(x, \theta) \in E(f_1, f_2)$, any path μ in M from x to x_0 lifts canonically to a path in $E(f_1, f_2)$ from (x, θ) to $(x_0, (f_1 \circ \mu)^{-1} * \theta * (f_2 \circ \mu))$ (here $*$ stands for successive travelling through paths, beginning with $(f_1 \circ \mu)^{-1}$). □

Remark 2.3. Given two coincidence points $x_1, x_2 \in M$ of f_1, f_2, their $\widetilde{g}$-values

$$\widetilde{g}(x_i) = (x_i, \text{ constant path at } f_1(x_i) = f_2(x_i)),$$

$i = 1, 2$, lie in the same path component of $E(f_1, f_2)$, precisely if the points x_1 and x_2 are *Nielsen equivalent*, i.e. they are joined by a path σ in M such that $f_1 \circ \sigma$ and $f_2 \circ \sigma$ are homotopic in N leaving endpoints fixed.

Now let φ be a virtual vector bundle over M; put $\widetilde{\varphi} := pr^*(\varphi)$.

Proposition 2.4. *(see* [HQ]*, 3.1). Assume that* M *is* $(k+1)$*-connected for some integer k. Then a choice of an orientation of* φ *at some point* $x_0 \in M$ *and of paths connecting* $f_1(x_0)$ *and* $f_2(x_0)$ *to some point* y_0 *in* N *determine an isomorphism*

$$i \ : \Omega_k^{fr}(\Lambda(N, y_0)) \longrightarrow \Omega_k(E(f_1, f_2); \widetilde{\varphi}) \ .$$

Proof. The choices induce isomorphisms

$$\Omega_k^{fr}(\Lambda(N, y_0)) \cong \Omega_k^{fr}(pr^{-1}(x_0)) \cong \Omega_k(pr^{-1}(x_0); \widetilde{\varphi}|)$$

and so does the fiber inclusion; this follows via a cell-by-cell argument applied to (projected) maps into M.

3. Homotopies and vector bundle isomorphisms

We will need the following well known fact in the construction of the invariant $\widetilde{\omega}(f_1, f_2)$ and for establishing its homotopy invariance and symmetry properties.

Lemma 3.1. *Let* P *be a CW-complex and* ξ *an* n*-dimensional vector bundle over a space* Q*. Then any homotopy* $H : P \times I \longrightarrow Q$ *between* $h_i = H(\ , i)$*,* $i = 0, 1$*, determines a vector bundle isomorphism* $h_0^*(\xi) \cong h_1^*(\xi)$ *over* P*, canonical up to regular homotopy.*

Indeed, the isomorphism bundle $\mathrm{Iso}(H^*(\xi), (h_0 \circ \text{first projection})^*(\xi))$ over $P \times I$ (with fiber $GL(n, \mathbb{R})$) has the required homotopy lifting property.

In particular, homotopies $F_i : M \times I \longrightarrow N$ from f_i to f_i', $i = 1, 2$, induce not only a fiber homotopy equivalence

$$E(f_1, f_2) \xrightarrow{\ \sim\ } E(f_1', f_2') \tag{3.2}$$

(which maps (x, θ) to $(x, (F_1(x, -))^{-1} * \theta * F_2(x, -)))$ but also a canonical isomorphism

$$\Omega_*(E(f_1, f_2); pr^*(\varphi)) \xrightarrow{\ \cong\ } \Omega_*(E(f_1', f_2'); pr'^*(\varphi')) \tag{3.3}$$

where $\varphi = f_1^*(TN) - TM$ (as in 1.3) and $\varphi' := f_{1*}'(TN) - TM$. This isomorphism remains unchanged when we deform the homotopies F_i while leaving them fixed on $M \times \{0, 1\}$.

Similarly the homeomorphism

$$E(f_1, f_2) \cong E(f_2, f_1), \quad (x, \theta) \longrightarrow (x, \theta^{-1})$$

and the tautological homotopy between $f_i \circ pr$, $i = 1, 2$, determine canonical isomorphisms

$$\begin{aligned} \Omega_*(E(f_1, f_2); pr^*(\varphi)) &\cong \Omega_*(E(f_1, f_2); pr^*(f_2^*(TN) - TM)) \\ &\cong \Omega_*(E(f_2, f_1); pr^*(f_2^*(TN) - TM)). \end{aligned} \tag{3.4}$$

Remark 3.5. Such symmetries are due to the paths occuring in $E(f_1, f_2)$ and to the resulting homotopy. In general, they are not available in the corresponding normal bordism groups of M. E.g. if TM and $f_1^*(TN)$ are orientable, but $f_2^*(TN)$ is not, then

$$\Omega_0(M; f_1^*(TN) - TM) \cong \mathbb{Z} \not\cong \mathbb{Z}_2 \cong \Omega_0(M; f_2^*(TN) - TM) \ .$$

4. The invariants $\widetilde{\omega}(f_1, f_2)$, $N(f_1, f_2)$ and $\omega(f_1, f_2)$.

In this section we construct and discuss our coincidence invariants. Moreover, we describe a technique to make Nielsen classes pathconnected. In particular, we prove theorems 1.9 and 1.10. We also sketch the definition of certain secondary obstructions in normal bordism.

If the map $(f_1, f_2) : M \longrightarrow N \times N$ is smooth and transverse to the diagonal

$$\Delta := \{(y, y) \in N \times N \mid y \in N\}$$

then the coincidence locus

$$C = C(f_1, f_2) := \{x \in M \mid f_1(x) = f_2(x)\} = (f_1, f_2)^{-1}(\Delta) \tag{4.1}$$

is a closed smooth $(m-n)$-dimensional manifold, equipped with the map

$$\widetilde{g} : C \longrightarrow E(f_1, f_2) \tag{4.2}$$

which sends $x \in C$ to $(x$, constant path at $f_1(x) = f_2(x))$, and with a stable tangent bundle isomorphism

$$\overline{g} : TC \oplus \widetilde{g}^*(pr^*(f_1^*(TN))) \cong \widetilde{g}^*(pr^*(TM)) \tag{4.3}$$

(since the normal bundle $\nu(\Delta, N \times N)$ of Δ in $N \times N$ is canonically isomorphic to the pullback of the tangent bundle TN under the first projection p_1).

If f_1 and f_2 are arbitrary continuous maps, apply the preceding construction to a smooth map (f_1', f_2') which approximates (f_1, f_2) and is transverse to Δ. Then there is a canonical isomorphism as in 3.3 induced by any sufficiently small homotopy from (f_1, f_2) to (f_1', f_2').

In any case the resulting triple $(C, \widetilde{g}, \overline{g})$ determines a well-defined normal bordism class

$$\widetilde{\omega}(f_1, f_2) \in \Omega_{m-n}(E(f_1, f_2); \widetilde{\varphi}) . \tag{4.4}$$

Arbitrary (possibly "large") homotopies of f_i yield a homotopy equivalence as in 3.2 and, in particular, a bijection of essential path components (cf. 1.8); indeed, the isomorphism 3.3 is compatible with $\widetilde{\omega}$. This proves the homotopy invariance of the Nielsen number claimed in 1.9 (i).

Similarly, $\widetilde{\omega}$ is compatible with the isomorphism 3.4 which, however, must first be composed with a suitable involution; this is needed since over $N \cong \Delta$ the composite of the obvious isomorphisms

$$TN \cong p_1^*(TN) \cong \nu(\Delta, N \times N) \cong p_2^*(TN) \cong TN \tag{4.5}$$

equals $-\mathrm{id}_{TN}$. The second claim in theorem 1.9 follows.

It remains to estimate the number of pathcomponents of the coincidence locus $C(f_1, f_2)$ (cf. 4.1) when f_1, f_2 are arbitrary maps (without smoothness or transversality assumptions). Embed N as a closed submanifold of $\mathbb{R}^{2n+1}$ and compose a linear homotopy in $\mathbb{R}^{2n+1}$ with a tubular neighbourhood projection onto N to obtain a map

$$\tau \; : \; U \times I \longrightarrow N \quad \text{such that} \quad \tau(y_0, y_1, i) \; = \; y_i, \quad i = 0, 1 \; ,$$

for all (y_0, y_1) in a suitable neighbourhood U of the diagonal Δ in $N \times N$. This allows us to extend the lifting $\widetilde{g}$ (cf. 1.4) to the map

$$\widetilde{G} \quad : \quad V := (f_1, f_2)^{-1}(U) \longrightarrow E(f_1, f_2) \; , \tag{4.6}$$

$\widetilde{G}(x) := (x, \tau(f_1(x), f_2(x), -)$, defined in the neighbourhood V of the coincidence locus $C(f_1, f_2) = (f_1, f_2)^{-1}(\Delta)$ in M. The decomposition of $E(f_1, f_2) = \cup \, A$ into (open!) pathcomponents yields a corresponding decomposition $V = \cup \, V_A$ which is compatible with any small (smooth, transverse) approximation (f_1', f_2') of (f_1, f_2). If $V_A \cap \, C(f_1, f_2) = \varnothing$, we may assume that the corresponding part of $C(f_1', f_2')$ is also empty. Even if $V_A \cap \; C(f_1, f_2) \neq \varnothing$, the corresponding part of $C(f_1', f_2')$ may be nonessential. Thus clearly

$$\#\pi_0(C(f_1, f_2)) \; \geq \; N(f_1', f_2') \; = \; N(f_1, f_2) \; .$$

Moreover, the compact manifold $C(f_1', f_2')$ has only finitely many pathcomponents.

The last claim in theorem 1.9 follows from

Proposition 4.7. *Assume* $n \neq 2$. *Given* (f_1, f_2), *there exists a pair of maps* (f_1', f_2') *and a homotopy* $(f_1, f_2) \sim (f_1', f_2')$ *relating each pathcomponent* $A \in \pi_0(E(f_1, f_2))$ (*with Nielsen class* $C_A \subset C(f_1, f_2)$, *cf.* 1.7, 2.3, *and* 3.2) *to a corresponding pathcomponent* $A' \in \pi_0(E(f_1', f_2'))$ (*with Nielsen class* $C_{A'} \subset C(f_1', f_2')$), *in such a way that*

(i) *each Nielsen class* $C_{A'} \subset C(f_1', f_2')$ *is pathconnected* (*or empty*)*;*
(ii) *if* C_A *is finite, then* $C_{A'}$ *consists of at most a single point; and*
(iii) *if* C_A *is empty, then so is* $C_{A'}$.

Proof. If $m < n$ or $N = \mathbb{R}$, then (f_1, f_2) is loose, and after a suitable homotopy every $C_{A'}$ is empty. The case $N = S^1$ will be discussed very explicitly in § 6 (at the end of the proof of theorem 1.13; also note: when $m \geq 2$ every *isolated* coincidence point x can be removed here by a deformation with small support near x). So we may assume that $m \geq n \geq 3$.

We will use the technique of the previous proof (cf. the lines following 4.6) to ensure that sufficiently small deformations do not cause new nonempty Nielsen classes. Thus, given $x \in C(f_1, f_2)$, we may assume that (f_1, f_2) is smooth and transverse to the diagonal $\Delta \subset N \times N$ near x; or, if the coincidence point x is isolated, then (in terms of local coordinates) $f_1 - f_2$ is linear on each ray starting from x. In either case in the end $C(f_1, f_2)$ and hence each Nielsen class C_A consists only of finitely many pathcomponents. We decrease their number by iterating the following procedure.

If the points $x_0, x_1 \in C_A$ lie in different pathcomponents join them by a path $\sigma : I \to M$ which otherwise avoids $C(f_1, f_2)$ and such that $f_1 \circ \sigma \sim f_2 \circ \sigma \;\; \mathrm{rel}\{0, 1\}$ via a homotopy $F : I \times I \to N$. Since $m \geq n \geq 3$ we may assume that σ and $f_1 \circ \sigma$ are smooth embeddings and that the transverse intersection of $F((\varepsilon, 1 - \varepsilon) \times (0, 1])$ with $f_1 \circ \sigma(I)$ consists of at most finitely many points of the form $F(t, s) =$

$f_1 \circ \sigma(t')$ such that $t \neq t'$. Now deform f_1, f_2 simultaneously near a small tubular neighbourhood $I \times B^{n-1}$ of $\sigma(I) = I \times \{0\}$ until $f_i(t, x') = f_i \circ \sigma(t)$, $i = 1, 2$, for all $t \in I$ and x' in the unit ball B^{n-1}; then replace $f_2(t, x')$ by $F(t, \|x'\|)$. This modification creates new coincidence points only along the whole arc $\sigma(I)$ and, in addition, possibly near its endpoints. We complete the proof of proposition 4.7 by applying the following result to suitable compact balls containing the arcs $f_1 \circ \sigma(I) = f_2 \circ \sigma(I)$ and $\sigma(I)$.

Lemma 4.8. *Given arbitrary continuous maps* $f_1, f_2 : M \to N$, *a compact* n*-ball* $Q \subset N$ *with collar, and a compact subset* $P \subset f_1^{-1}(Q) \cap f_2^{-1}(Q) \subset M$, *there exist homotopic maps* $f_1' \sim f_1$ *and* $f_2' \sim f_2$ *with coincidence locus*

$$C(f_1', f_2') \;=\; C(f_1, f_2) \;\cup\; P \;.$$

If, in addition, P *is a compact* m*-ball in* M *with collar, then after a further homotopy* $C(f_1', f_2')$ *can be made to be homeomorphic to the quotient space* $(C(f_1, f_2) \cup P)/P$.

(*No restrictions are imposed here on* $m = \dim M$ *and* $n = \dim N$).

Proof. The assumption on Q means that Q is the image of the compact n-ball $B^n(1) \subset \mathbb{R}^n$ of radius 1 under a homeomorphism $B^n(1 + \varepsilon) \cong \widehat{Q} \subset N$ for some $\varepsilon > 0$. Let $* \in Q$ be the centre point and let Cone $(Q) \subset N \times I$ denote the cone with basis $Q \times \{0\}$ and top vertex $z_0 := (*, 1)$. By shrinking Q and stretching the collar radially we can construct a continuous family (h_s), $s \in [0, 1)$, of selfhomeomorphisms of N which starts with $h_0 = \mathrm{id}$ and converges to a selfmap h_1 of N taking $N - Q$ homeomorphically to $N - \{*\}$ and such that $h_1(Q) = \{*\}$.

Next choose a continuous function $\delta : M \to [0, 1]$ satisfying $\delta^{-1}(\{1\}) = P$. The required maps f_i' can be defined by

$$f_i'(x) \;=\; h_{\delta(x)}(f_i(x)) \;, \quad x \in M, \quad i = 1, 2 \;.$$

If $P \subset M$ is also an m-ball with collar, construct a similar homotopy $(\overline{h}_s)$ of selfmaps of M. Then the maps f_i' factor through $\overline{h}_1$, $f_i' = f_i'' \circ \overline{h}_1 (\sim f_i'' \circ \overline{h}_0 = f_i''$, $i = 1, 2$, and $C(f_1'', f_2'') = (C(f_1, f_2) \cup P)/P$. □

Clearly the invariants $\omega(f_1, f_2)$ (defined in [K 3]; see also 1.2) and $\widetilde{\omega}(f_1, f_2)$ (cf. 4.4) are related by the equation

$$\omega(f_1, f_2) \;=\; pr_*(\widetilde{\omega}(f_1, f_2)) \tag{4.9}$$

where the homomorphism

$$pr_* \;:\; \Omega_*(E(f_1, f_2); \widetilde{\varphi}) \;\longrightarrow\; \Omega_*(M; \varphi) \tag{4.10}$$

is induced by the fiber projection pr (cf. 1.4, 1.3, and 1.6). In general it leads to a loss not only of information but also of symmetry: while $\widetilde{\omega}(f_1, f_2)$ and $\widetilde{\omega}(f_2, f_1)$ are always equally strong, the invariants $\omega(f_1, f_2)$ and $\omega(f_2, f_1)$ need not be (compare remark 3.5).

In the very special case $f_1 = f_2 =: f$ of example II pr has a global section s such that $\widetilde{\omega}(f,f)$ equals $s_*(\omega(f,f))$ and hence is precisely as strong as the selfcoincidence invariant $\omega(f)$ studied in [K 3].

In general, however, it may happen that $\omega(f_1,f_2) = 0$, but $\widetilde{\omega}(f_1,f_2)$ gives rise to interesting secondary obstructions of the form $h(\widetilde{\omega}(f_1,f_2))$, where h is a well-defined homomorphism on the kernel of pr_*. E.g. if M and N are stably parallelizable (and hence $\ker pr_*$ lies in the framed bordism group $\Omega_*^{fr}(E(f_1,f_2))$), we may construct a homomorphism

$$h_2 \; : \; \ker pr_* \; \longrightarrow \; \widetilde{\Omega}^{fr}_{*+1}(N)/(f_{1*} - f_{2*})(\Omega^{fr}_{*+1}(M)) \tag{4.11}$$

in the spirit of 2.1 as follows. Given $c \in \ker pr_*$, pick a framed singular manifold $C \xrightarrow{\widetilde{g}=(g,\theta)} E(f_1,f_2)$ representing c and a framed nulbordism $G : B \to M$ of (C,g); define $h_2(c)$ to be the class of the closed framed manifold

$$-B \; \cup_{\partial B = C\times\{0\}} \; C \times I \; \cup_{C\times\{1\}=\partial B} \; B$$

together with the map $f_1{\circ}G \cup \theta \cup f_2{\circ}G$ into N. If in addition $N = S^n, n \geq 2$, then the secondary obstruction $h_2(\widetilde{\omega}(f_1,f_2))$ lies in a quotient of the reduced bordism group $\widetilde{\Omega}^{fr}_{m-n+1}(S^n) \cong \Omega^{fr}_{m-2n+1} \cong \pi^S_{m-1-2(n-1)}$ and generalizes the invariant $\widetilde{\omega}_2(f_1,f_2)$ discussed in theorem 1.14 and section 8. □

Next we turn to the

Proof of theorem 1.10. We proceed in close analogy to the proof of theorem 3.1 in [K 1], p. 37–39.

We may assume that $(f_1,f_2) : M \to N \times N$ is smooth and transverse to the diagonal Δ. Given an inessential pathcomponent A of $E(f_1,f_2)$, let $(\mathfrak{C}, \widetilde{G}, \overline{G})$ be a nulbordism of the corresponding triple $(C_A, \widetilde{g}|C_A, \overline{g}|\,)$ (cf. 1.7 and 1.8). We will use the three data $\mathfrak{C}$, $\widetilde{G}$ and $\overline{G}$ to "seal off" the coincidence manifold C_A without creating new coincidences.

First we identify $\mathfrak{C}$ with a smooth submanifold of $M \times I$ such that

$$\mathfrak{C} \cap \partial(M \times I) \; = \; \partial\mathfrak{C} \; = \; C_A \; \subset \; M \times \{0\} \; .$$

For this purpose we use an embedding

$$i \; = \; (i_1, i_2) \; : \; \mathfrak{C} \; \hookrightarrow \; M \times I$$

where $i_1 : \mathfrak{C} \hookrightarrow M$ is (a deformation of) $\mathrm{pr} \circ \widetilde{G}$ (compare 1.4) and avoids all other coincidence components $C_{A'}, A' \neq A$; on a collar $C_A \times [0, \frac{1}{2}]$ of C_A in $\mathfrak{C}$ the function i_2 is essentially defined by the second projection, and it takes the constant value $\frac{1}{2}$ outside of this collar (compare figure 3.8 in [K 1]).

The second nulbordism datum $\widetilde{G}$ yields also a map $\Theta : \mathfrak{C} \times I \to N$ or, equivalently, homotopies

$$H_j : \quad f_j{\circ}i_1 \sim \tau \;\; : \;\; \mathfrak{C} \longrightarrow N \; , \qquad j = 1,2,$$

where $\tau(x) := \Theta(x, \frac{1}{2})$ for $x \in \mathfrak{C}$.

Our third datum $\overline{G}$, together with the homotopy H_1, induces a (stable) vector bundle isomorphism

$$T\mathfrak{C} \oplus \tau^*(TN) \;\cong\; T(M \times I)|\mathfrak{C}$$

(compare 4.3). We can destabilize it to obtain a (nonstable) isomorphism

$$\overline{G}^{\#} \;:\; \tau^*(TN) \cong (\tau,\tau)^*(\nu(\Delta, N \times N)) \;\cong\; \nu(\mathfrak{C}, M \times I)$$

of normal bundles which extends the isomorphism

$$f_1^*(TN)|C_A \;\cong\; ((f_1,f_2)|C_A)^*(\nu(\Delta, N \times N)) \;\cong\; \nu(C_A, M)$$

induced by $(f_1, f_2) : M \to N \times N$.

This (and the homotopies H_1, H_2) allow us to extend the maps (f_1, f_2) and $(\tau, \tau) : \mathfrak{C} \to N \times N$ to the union of $M = M \times \{0\}$ with a tubular neighbourhood of $\mathfrak{C}$ in $M \times I$ (and even with a neighbourhood of the "shadow"

$$R_1 \;= \{(i_1(x), t) \in M \times I | x \in \mathfrak{C},\; 0 \le t \le i_2(x)\} \;),$$

without creating any new coincidences outside of $\mathfrak{C}$. In analogy to [K 1], p. 39, this procedure yields a deformation of (f_1, f_2) which makes all inessential Nielsen coincidence classes empty. Applying proposition 4.7 completes the proof.

Note that we have used the dimension assumption $m < 2n-2$ at three separate occasions: (i) when approximating $\mathrm{pr} \circ \widetilde{G}$ by an embedding into M; (ii) when destabilizing $\overline{G}$; and, above all, (iii) in order to make sure that no new coincidences occur on the shadow R_1 outside of $\mathfrak{C}$. □

The dimension restriction in theorem 1.10 turns out to be unnecessary in a number of cases. For a better understanding of one such case in § 6 it is helpful to study precompositions.

Proposition 4.12. *Let L be a smooth closed connected manifold and $e : L \to M$ a map. Assume that in the diagram*

$$\pi_1(L, \ell_0) \;\xrightarrow{e_*}\; \pi_1(M, x_0 := e(\ell_0)) \;\xrightarrow{(f_1,f_2)_*}\; \pi_1(N \times N, (f_1, f_2)(x_0))$$

$(f_1, f_2)_* \circ e_*$ and $(f_1, f_2)_*$ have equal images.

Then $N(f_1 \circ e, f_2 \circ e) \;\le\; N(f_1, f_2)$.

Proof. In view of 1.9 (i) we need only consider the case where (f_1, f_2) and e are smooth and transverse to the diagonal Δ and to the coincidence locus $C(f_1, f_2) = (f_1, f_2)^{-1}(\Delta)$, resp. (cf. 4.1), and where in addition $N(f_1 \circ e, f_2 \circ e) > 0$ and $x_0 \in C(f_1, f_2)$. Then it follows from our assumption and from 2.1 that the map

$$E(f_1 \circ e, f_2 \circ e) \;\longrightarrow\; E(f_1, f_2)$$

determined by e makes each path component A of $E(f_1, f_2)$ correspond to a unique path component A' of $E(f_1 \circ e, f_2 \circ e)$. Now, if A is inessential then so is A'. Indeed, let C_A and $C_{A'}$ denote the corresponding parts of the coincidence manifolds $C(f_1, f_2)$ and $C(f_1 \circ e, f_2 \circ e) = e^{-1}(C(f_1, f_2))$. Then a nulbordism of C_A yields a nulbordism of $C_{A'}$ by transverse intersection with e in M. □

For later use we note also one of the many interrelations with degree theory (compare 1.11).

Proposition 4.13. *If* N *is a Lie group (and hence there is a multiplication of maps into* N*), then*

$$N(f_1, f_2) \;=\; N(f_1 \cdot f_2^{-1},\; \textit{constant map}\;)$$

and this Nielsen number equals 0 *or* $\#(\pi_1(N)/(f_{1*} - f_{2*})(\pi_1(M)))$. *Moreover,*

$$MCC(f_1, f_2) \;=\; MCC(f_1 \cdot f_2^{-1},\; \textit{constant map}\;)\quad .$$

Proof. The diffeomorphism

$$(N \times N, \Delta) \longrightarrow (N \times N, 1 \times N)\;,$$

$(y_1, y_2) \longrightarrow (y_1 \cdot y_2^{-1}, y_2)$, and the natural homeomorphism $E(f_1, f_2) \cong E(f_1 \cdot f_2^{-1}, 1)$ lead to an isomorphism of normal bordism groups which takes $\widetilde{\omega}(f_1, f_2)$ to $\widetilde{\omega}(f_1 \cdot f_2^{-1}, 1) = \widetilde{\deg}(f_1 \cdot f_2^{-1})$. In particular, the corresponding Nielsen numbers agree. Also the transitive action of $\pi_1(N)$ on $\pi_0(E(f_1 \cdot f_2^{-1}, 1))(\approx \operatorname{coker}(f_1 \cdot f_2^{-1})_*$, cf. 2.1) is compatible with $\widetilde{\deg}(f_1 \cdot f_2^{-1})$. Hence either all pathcomponents of $E(f_1, f_2)$ are essential or all are inessential.

If the pairs of maps $(f_1 \cdot f_2^{-1}, 1)$ and (g_1, g_2) are homotopic, then so are (f_1, f_2) and $(f_1 \cdot f_2, g_2 \cdot f_2)$. Since (f_1, g_2) and $(g_1 \cdot f_2, g_2 \cdot f_2)$ have the same coincidence set we conclude that $MCC(f_1, f_2) \leq MCC(f_1 \cdot f_2^{-1}, 1)$. The full equality follows similarly.

5. Classical Nielsen coincidence theory

In this section we consider the case when M and N have the same dimension. Then

$$\widetilde{\omega}(f_1, f_2) \;=\; \{\widetilde{\omega}_A(f_1, f_2)\} \;\in\; \oplus_{A \in \pi_0(E(f_1, f_2))}\, \Omega_0(A; \widetilde{\varphi}|A)$$

(cf. 1.5 and 1.7).

Now, a 0-dimensional normal bordism group of any pathconnected (and locally pathconnected) space is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}_2$ according as the coefficient bundle is orientable or not, and this is usually expressed by the first Stiefel-Whitney class. In the situation at hand, $\widetilde{\varphi}|A = (pr|A)^*(\varphi)$ is orientable if and only if

$$w_1(\varphi) \;=\; w_1(M) + f_1^*(w_1(N)) \;\in\; H^1(M; \mathbb{Z}_2) \tag{5.1}$$

(cf. 1.3 and 1.6) vanishes when evaluated on the image of the fundamental group of A under pr_*. In view of proposition 2.1 we obtain

Proposition 5.2. *Let* $x_0 \in M$ *be a coincidence point with* $y_0 := f_1(x_0) = f_2(x_0)$. *Given a pathcomponent* A *of* $E(f_1, f_2)$, *pick* $[\theta]$ *in* $\pi_1(N, y_0)$ *such that* $(x_0, \theta) \in A$ *(i.e.* $A = A_\theta$ *corresponds to the class of* $[\theta]$ *in the Reidemeister set).*

Then $\Omega_0(A; \widetilde{\varphi}|A) \cong \mathbb{Z}$ *if and only if*

$$w_1(M)(\mu) \;=\; f_1^*(w_1(N))(\mu)$$

for all $\mu \in \pi_1(M, x_0)$ *such that* $f_{2*}(\mu) = [\theta]^{-1} \cdot f_{1*}(\mu) \cdot [\theta]$. *Otherwise* $\Omega_0(A; \widetilde{\varphi}|A) \cong \mathbb{Z}_2$.

If $\pi_1(N, y_0)$ is abelian, then this criterion is independent of the pathcomponent A and we have

$$\Omega_0(E(f_1, f_2); \widetilde{\varphi}) \cong \begin{cases} \oplus_A \mathbb{Z} & \text{if } w_1(\varphi) \equiv 0 \text{ on } \ker(f_{2*} - f_{1*}) \ ; \\ \oplus_A \mathbb{Z}_2 & \text{else} \quad . \end{cases}$$

However, if $\pi_1(N, y_0)$ is not commutative, then both $\mathbb{Z}$ and $\mathbb{Z}_2$ may occur as direct summands (possibly corresponding to inessential Nielsen classes), e.g. in some cases when M is the Klein bottle and N is the punctured torus. □

Now consider the generic situation where (f_1, f_2) is smooth and transverse to the diagonal Δ in $N \times N$. Then a coincidence point x can contribute a well-defined *integer* index ± 1 (the intersection number of (f_1, f_2) with Δ at x) to $\widetilde{\omega}(f_1, f_2)$ if and only if $w_1(\varphi)$ vanishes on all $\mu \in \pi_1(M, x)$ such that $f_{1*}(\mu) = f_{2*}(\mu)$ (the sign depends then on the choice of an orientation of $\widetilde{\varphi}$ on the path component of (x, constant path at $f_1(x) = f_2(x)$). E.g. in fixed point theory (where $f_1 = \mathrm{id}$ and hence $\varphi = 0$), or, more generally, if f_1 is *orientation true* (cf. [BGZ], p. 49), i.e. if we can (and do) choose an orientation already of φ (not just of $\widetilde{\varphi}|A$), we can attach well-defined *integer* indices to all coincidence points.

Whether indices lie in $\mathbb{Z}$ or in $\mathbb{Z}_2$, $\widetilde{\omega}(f_1, f_2)$ assigns to every pathcomponent A of $E(f_1, f_2)$, i.e. to every Nielsen class (cf. 2.3) of coincidence points, its (total) index in a very natural way, and our definition of Nielsen numbers coincides with the classical one (e.g. as in [BGZ], 3.5 and 3.6) at least if M and N are orientable.

In case $M = N$ has dimension at least 3 and f is a selfmap of M we can easily modify the proof of theorem 1.10 (using proposition 4.7) to show that $N(f, \mathrm{id})$ agrees with the minimum number

$$MC(f, \mathrm{id}) \ := \ \min\{\#C(f_1', f_2') | f_1' \sim f, f_2' \sim \mathrm{id}\}$$

of coincidence *points* (not just pathcomponents) or, by [BR], with the minimum number

$$MF(f) \ := \ \min\{\#C(f', \mathrm{id}) | f' \sim f\}$$

of fixed points. This is the classical Wecken theorem for closed smooth manifolds (cf. [B], p. 12).

This section used the language of 0-dimensional bordism theory. Techniques for calculating normal bordism groups also in dimensions $m - n = 1, 2, 3$ can be found e.g. in [K 1], § 9.

6. The case when N is 1-dimensional

Because of the dimension restriction $m < 2n - 2$ theorem 1.10 is uninteresting for low n. In this section we will study the case $n = 1$ by different methods and answer our original question explicitly.

If $N = \mathbb{R}$ there is no problem: obviously f_1 and f_2 are homotopic to different (and hence coincidence free) constant maps.

For the remainder of this section let N be the circle S^1. Now S^1 enjoys three very special properties. First of all, it is a Lie group and we can use proposition 4.13. As a consequence we need to prove theorem 1.13 only for pairs of the form

$(f,*)$ where $*$ stands for a constant map with value, say, $y_0 = f(x_0)$ for some $x_0 \in M$. Let d be the nonnegative integer characterized by the equation

$$f_*(\pi_1(M,x_0)) \;=\; d\cdot\pi_1(S^1,y_0) \;\cong\; d\mathbb{Z}.$$

Then, on one hand, we have

$$N(f,*) \;\le\; d$$

since we can index *all* the path components of $E(f,*)$ (essential or not) by the Reidemeister set $R(f,*,x_0) = \operatorname{coker} f_* \cong \mathbb{Z}/d\mathbb{Z}$ (cf. prop. 2.1). On the other hand, if we pick a loop $e : (S^1,1) \to (M,x_0)$ such that $f\circ e : S^1 \to S^1$ has degree d, we obtain

$$d \;=\; N(f\circ e,*) \;\le\; N(f,*)\ .$$

Indeed, the inequality to the right follows from proposition 4.12; moreover, $f\circ e$ is homotopic to the standard selfmap of the circle of degree d, whose d roots clearly lie in pairwise distinct Nielsen classes (compare 2.3). This yields one possible proof of the first claim in theorem 1.13.

An alternative proof and the third claim follow from another special property of the circle: S^1 is an Eilenberg–MacLane–space for $H^1(\ ;\mathbb{Z})$. Thus the homotopy class of f is already determined by $f_* : H_1(M;\mathbb{Z}) \to H_1(S^1;\mathbb{Z})$. Also we obtain the indicated isomorphisms in the commuting diagram

$$\begin{array}{ccc}
[M,S^1] & \xrightarrow{\;\cong\;} & H^1(M,\mathbb{Z}) \\
\big\downarrow \text{deg} & & \big\downarrow \wr\wr\; PD \\
\Omega_{m-1}(M;-TM) & \xrightarrow{\;\mu\;} & H_{m-1}(M;\widetilde{\mathbb{Z}}_M)
\end{array} \quad .$$

Finally recall that S^1 is also a Thom space. Thus we can describe a deformation of a generic map $f : M \to S^1$ by exhibiting a bordism of the cooriented 1-codimensional submanifold $C = f^{-1}\{*\}$ of M.

Assume that $m \ge 2$ and that there are different pathcomponents C_0 and C_1 of C belonging to the same Nielsen class (cf. 2.3). They can be joined by a smoothly immersed curve $\sigma : [0,1] \to M$ which is transverse to C and such that $f\circ\sigma$ is nulhomotopic rel$\{0,1\}$. Let the crossings of σ with C have opposite signs at $\sigma(0)$ and $\sigma(1)$. Then the intersection $\sigma(I)\cap C$ consists of an equal number of positive and negative crossings. As we go along the curve σ there may be consecutive crossings $\sigma(t),\sigma(t') \notin \{\sigma(0),\sigma(1)\}$ with opposite signs. If they lie in the same pathcomponent C' of C pick a path in C' joining them and replace $\sigma|(t-\varepsilon,t'+\varepsilon)$ by the corresponding "parallel" path just outside of C'. In the end there must remain consecutive crossings $\sigma(t),\sigma(t')$ (with opposite signs) which lie in different pathcomponents C_0' and C_1' of C. Apply connected sum surgery along the arc $\sigma([t,t'])$ (which can be made embedded via suitable shortcuts). This corresponds to a homotopy of f which decreases $\#\pi_0(f^{-1}\{*\})$. Iterating this procedure we see that $MCC(f,*) \le N(f,*)$ if $N(f,*) \ne 0$. In view of 1.9 (iii) this completes the proof of theorem 1.13. □

Proposition 6.1. *Given maps $f_1,f_2 : M \to S^1$ and a coincidence point $x_0 \in M$, put $y_0 := f_1(x_0) = f_2(x_0)$. Let the kernel of*

$$f_{1*} \;-\; f_{2*} \;:\; \pi_1(M,x_0) \;\longrightarrow\; \pi_1(S^1,y_0)$$

act in the standard fashion on the universal covering space $(\widetilde{M}, \widetilde{x}_0)$ *of* (M, x_0) *by covering transformations.*

Then the fiber space $E(f_1, f_2)$ *is fiber homotopically equivalent to*

$$(\widetilde{M}/\ker(f_{1*} - f_{2*})) \ \times \ \operatorname{coker}(f_{1*} - f_{2*})$$

(*where the cokernel has the discrete topology*).

If the maps f_1, f_2 *are homotopic (e.g. when they are coincidence free), then* $E(f_1, f_2)$ *is fiber homotopically equivalent to* $M \times \mathbb{Z}$.

Proof. Since S^1 is a Lie group, $E(f_1, f_2)$ is fibre homeomorphic to $E(f := f_1 \cdot f_2^{-1}, 1)$ and hence to the pullback, under f, of the starting point fibration of

$$P(S^1, 1) \ = \ \{\theta : ([0,1], 1) \longrightarrow (S^1, 1) \text{ continuous}\}.$$

Since every path θ can be deformed $\operatorname{rel}\{0,1\}$ to a path with constant speed, $P(S^1, 1)$ is fiber homotopically equivalent to the universal covering space of S^1. Clearly, its pullback under f allows the indicated description in terms of $\widetilde{M}$, provided a coincidence point exists. This is always the case after a suitable homotopy (cf. also 3.2).

7. The setting of homotopy groups

In this section we study the situation where $M = S^m$. Moreover, we assume $n \geq 2$ (having settled the case $n = 1$ in § 6).

Given basepoint preserving maps $f_1, f_2 : (S^m, x_0) \to (N, y_0)$, a fixed choice of local orientations of S^m and N at x_0 and y_0, resp., determines an isomorphism

$$i \ : \ \Omega^{fr}_{m-n}(\Lambda(N, y_0)) \ \longrightarrow \ \Omega_{m-n}(E(f_1, f_2); \widetilde{\varphi}) \tag{7.1}$$

(apply 2.4 using constant paths). For any bordism class $z = [C, \widetilde{g}, \overline{g}]$ in the target the corresponding class $i^{-1}(z)$ can be described as follows. Pick any homotopy $G : C \times [0,1] \to S^m$ joining $g := pr \circ \widetilde{g}$ to the constant map at x_0 and lift it in a natural way to a homotopy $\widetilde{G}$ in $E(f_1, f_2)$ joining $\widetilde{g}$ to $(x_0, \widetilde{g}')$, where for $x \in C$ the loop $\widetilde{g}'(x)$ is the composite of paths

$$y_0 \xrightarrow{(f_1 \circ G(x,-))^{-1}} f_1(g(x)) \xrightarrow{\theta(\widetilde{g}(x))} f_2(g(x)) \xrightarrow{f_2 \circ G(x,-)} y_0 \ . \tag{7.2}$$

$\widetilde{G}$ also induces a vector bundle isomorphism joining $\overline{g}$ to a framing $\overline{g}'$ of C. Then

$$i^{-1}([C, \widetilde{g}, \overline{g}]) \ = \ [C, \widetilde{g}', \overline{g}']. \tag{7.3}$$

We use isomorphisms such as i to identify all $\widetilde{\omega}$-invariants with elements in the *same* group which no longer varies with f_1 or f_2. This identification is compatible with base point preserving homotopies (compare 3.2 and 3.3). Thus we obtain well-defined maps

$$\begin{array}{ccc} \pi_m(N, y_0) \times \pi_m(N, y_0) & \xrightarrow{\ \widetilde{\omega}\ } & \\ & & \Omega^{fr}_{m-n}(\Lambda(N, y_0)) \\ \pi_m(N, y_0) & \xrightarrow{\ \widetilde{\deg} = \widetilde{\omega}(-, y_0)\ } & \end{array} \ . \tag{7.4}$$

Proposition 7.5. (i) *For all* $[f_1], [f_2] \in \pi_m(N, y_0)$

$$\widetilde{\omega}(f_1, f_2) \;=\; \widetilde{\deg}(f_1) + (-1)^n \operatorname{inv}_*(\widetilde{\deg}(f_2))$$

where inv_* *is induced by the involution on* $\Lambda(N, y_0)$ *which inverses the direction of a loop* (*and possibly by canonical reframings in case* N *is not stably parallelizable*)*;*

(ii) $\widetilde{\deg}$ *is a group homomorphism.*

Proof. After a base point preserving homotopy and a further small approximation we may assume that f_1 and f_2 take different constant values (which lie near y_0) on complementary halfspheres $S^m_\pm$, i.e.

$$f_1|S^m_+ \;\equiv\; * \;\neq\; *' \;\equiv\; f_2|S^m_-.$$

Then clearly

$$\widetilde{\omega}(f_1, f_2) \;=\; \widetilde{\omega}(f_1, *') \;+\; \widetilde{\omega}(*, f_2) \;=\; \widetilde{\omega}(f_1, y_0) \;+\; \widetilde{\omega}(y_0, f_2)$$

and our first claim follows from 4.5 and 7.2. (If N is not stably parallelizable the process of interchanging f_2 with y_0 in $\widetilde{\omega}(y_0, f_2)$ and $\widetilde{\omega}(f_2, y_0) = \widetilde{\deg}(f_2)$ involves also the reframing induced by the evaluation map $C \times I \to N$; compare 3.1, 7.2, and 7.3).

A very similar argument implies the additivity of the refined degree $\widetilde{\deg}$. □

In view of formula 7.5 (i) it is natural to expect some interrelation between the following two possible conditions concerning S^m and N.

Condition A. For any two maps $f_1, f_2 : S^m \to N$ the Nielsen number (or, equivalently, the $\widetilde{\omega}$-invariant) is the only looseness obstruction; in other words: (f_1, f_2) is loose if and only if $N(f_1, f_2) = 0$.

Condition B. The refined degree homomorphism $\widetilde{\deg}$ in diagram 7.4 is injective.

Proposition 7.6. (i) *Assume that* N *allows a self-map a without fixed points* (*this holds e.g. if* N *has a nowhere vanishing vector field*)*. Then condition B implies condition A* (*and more specifically that* (f_1, f_2) *is loose if and only if* f_2 *is homotopic to* $a \circ f_1$)*.*

(ii) *Assume that* $\pi_m(N - \{point\}) = 0$*. Then condition A implies condition B.*

Proof. (i) The assumption implies that for every map $f_1 : S^m \to N$ there is a map $\widehat{f}_2$ (e.g. $\widehat{f}_2 = a \circ f_1$) which has no coincidences with f_1; this is essentially all we need in the proof.

Now consider any pair (f_1, f_2) with vanishing Nielsen number. After suitable homotopies f_1, f_2 and $\widehat{f}_2$ preserve base points. Now $\widetilde{\omega}(f_2, f_1) = \widetilde{\omega}(\widehat{f}_2, f_1) = 0$ and hence $\widetilde{\deg}(f_2) = \widetilde{\deg}(\widehat{f}_2)$ by 7.5 (i). If condition B holds f_2 and $\widehat{f}_2$ are homotopic and (f_1, f_2) is loose.

(ii) Suppose that condition A holds. Given $[f] \in \ker(\widetilde{\deg})$, the pair $(f_1, f_2) := (f, y_0)$ has Nielsen number zero and must be loose. Then, by [Br], f is homotopic to a map which avoids y_0. In view of the assumption in (ii), $[f] = 0$ and $\widetilde{\deg}$ is injective. □

8. Maps between spheres

In this section we work out the details of Example IV of the introduction. Thus let N be the n-sphere (with the antipodal selfhomeomorphism denoted by a). This setting provides a good testing ground for the strength of our methods. Indeed, given *any* space M, the answer to the looseness question for maps $f_1, f_2 : M \to S^n$ is known from the very beginning (cf. [DG], 1.10): if (f_1, f_2) is homotopic to a coincidence-free pair (f_1', f_2') then $f_1' \sim a \circ f_2'$ via the homotopy $((1-t)f_1' - tf_2')/\|((1-t)f_1' - tf_2')\|$; thus (f_1, f_2) is loose if and only if $f_1 \sim a \circ f_2$.

Now assume also that $M = S^m$ and $n \geq 2$.

First we want to make $C(f_1, f_2)$ pathconnected. After suitable homotopies our maps have the form described in the proof of proposition 7.5, with $*'$ and $*$ being regular values of f_1 and f_2, resp.; moreover f_1 is defined, on a tubular neighbourhood $f_1^{-1}\{*'\} \times B^n$ of $f_1^{-1}\{*'\}$, by the projection to $B^n/\partial B^n \cong S^n$ (as in the Pontryagin-Thom construction), and similarly f_2. Then any two pathcomponents of $C(f_1, f_2) = f_1^{-1}\{*'\} \cup f_2^{-1}\{*\}$ can be joined by an arc $\sigma(I)$ in S^m which intersects their tubular neighbourhoods in appropriate rays and such that $f_1(\sigma(I))$ and $f_2(\sigma(I))$ lie in the same arc from $*$ to $*'$ in S^n. Thus the techniques of the proof of proposition 4.7 allow us to decrease the number of pathcomponents of $C(f_1, f_2)$ successively (even when $n = 2$). This proves the last claim in theorem 1.14.

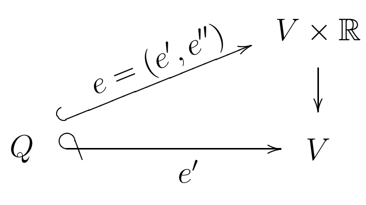
Next apply the discussion of § 7 to the case $N = S^n$. Then by proposition 7.6 condition A is equivalent to the injectivity of $\widetilde{\deg}$. Most of theorem 1.14 as well as corollary 1.15 follow as soon as we have established the indicated connection with the James-Hopf invariants.

For this purpose we will exploit the close relationship between loop spaces and immersions. Recall the bijection

$$\mathcal{J}_1(V, \varepsilon^{n-1}) \xrightarrow{\beta \circ \iota} [SV_c, S^n] \cong [V_c, \Lambda(S^n, y_0)] \tag{8.1}$$

discussed in [KS], theorem 1.2 and example (i) (on p. 284). Here V and $(S)V_c$, resp., denote an arbitrary smooth manifold without boundary and (the suspension of) its one-point compactification, resp.. $\mathcal{J}_1(V, \varepsilon^{n-1})$ is the set of bordism classes of embeddings $e : Q \hookrightarrow V \times \mathbb{R}$ which project to framed $(n-1)$-codimensional immersions in V; the commuting diagram

(8.2)



subsumes this projection property. By forgetting it we obtain a *bijection* ι onto the bordism set of framed codimension-n embeddings in $V \times \mathbb{R}$. The Pontryagin-Thom procedure then yields a further bijection β onto the indicated base point preserving homotopy set.

Now, given any bordism class in $\Omega_*^{fr}(\Lambda(S^n, y_0))$, represent it by a singular manifold (V, v) and apply the inverse bijection $(\beta \circ \iota)^{-1}$ (cf. 8.1) to the homotopy class of the map v, to obtain an immersion $e' : Q \looparrowright V$ as in 8.2.

Proposition 8.3. *This procedure yields a well-defined isomorphism from the framed bordism group* $\Omega_r^{fr}(\Lambda(S^n, y_0))$, $r \in \mathbb{Z}$, *onto the joint bordism group* $I_{r,n-1}$ *of quadruples* (e', Q, V, e'') *where*

(i) $e' : Q \looparrowright V$ *is a framed* $(n-1)$*-codimensional immersion between closed smooth manifolds;*
(ii) V *is framed and has dimension* r*; and*
(iii) $e'' : Q \to \mathbb{R}$, *together with* e', *determines an embedding* $e = (e', e'')$ *(which "decompresses"* e'*) as in* 8.2.

A *joint bordism* of such quadruples is free to alter Q and V simultaneously (compare [S]).

Proof. Given a bordism (W, w) between, say, (V_1, v_1) and (V_2, v_2), we can replace (V, v) in the above-mentioned construction by the closed singular manifold $(W \cup_\partial -W, w \cup w)$ and obtain the required joint bordism between the decompressed immersions which correspond to (V_1, v_1) and (V_2, v_2). Thus our construction yields a well-defined inverse of the obvious homomorphism induced by such forgetful maps as ι in 8.1. □

Next, given a quadruple as in 8.3, we analyze the multiple self-intersections of e'. After a small deformation we may assume that they are all transverse. Then for $k \geq 1$ the set of k-tuple points

$$(8.4)\quad Q_k := \{(x_1, .., x_k) \in Q^k | e'(x_1) = .. = e'(x_k), e''(x_1) < e''(x_2) < .. < e''(x_k)\}$$

is a closed manifold, smoothly immersed in V with a natural framing (inherited from the framing of e' and the ordering of the intersection branches given by e''). Using the stable parallelization of V and forgetting the immersion, we obtain the stable bordism class

$$(8.5)\qquad h_{k+1}([e', Q, V, e'']) \ := \ [Q_k] \ \in \ \Omega^{fr}_{r-k(n-1)} \cong \pi^S_{r-k(n-1)}$$

(note the shift of the index; e.g. the h_2-value equals $[Q]$). Also we put

$$h_1([e', Q, V, e'']) := [V] \ .$$

Proposition 8.6. *This construction determines a well-defined isomorphism*

$$h = \bigoplus_{k \geq 1} h_k \ : \ I_{r,n-1} \xrightarrow{\cong} \bigoplus_{k \geq 1} \pi^S_{r-(k-1)(n-1)}$$

(*compare* 8.3).

Proof. Joint bordisms (and disjoint unions) yield bordisms (and unions) of self-intersections; thus h is a well-defined homomorphism.

As for bijectivity, we follow an unpublished argument of Paul Schweitzer (compare also [K 2], pp. 81–83). Given $[e', Q, V, e''] \in \ker h$, we may assume that e' is *self-transverse*, i.e. all self-intersections are transverse. Let k be the highest occuring multiplicity. Then Q_k is an embedded submanifold of V with a tubular neighbourhood of the form $Q_k \times (\mathbb{R}^{n-1})^k$ which intersects $e'(Q)$ in the branches $Q_k \times (\mathbb{R}^{n-1})^j \times \{0\} \times (\mathbb{R}^{n-1})^{k-j-1}$, $j = 0, \ldots, k-1$. Now pick a nulbordism of

Q_k and glue its product with the unit ball B_k of $(\mathbb{R}^{n-1})^k$ to $V \times [0,1]$ along $Q_k \times B_k \subset V \times \{1\}$. The resulting joint bordism "seals off" Q_k so that only self-intersections of strictly smaller multiplicity survive. Iterating this procedure we obtain a nulbordism of (e', Q, V, e''). Thus h is injective.

Surjectivity follows in a similar way. Given a closed framed manifold Q'_k of dimension $r-k(n-1)$, equip the product $Q'_k \times B_k$ with the obvious codimension-$(n-1)$ immersion as described above, seal off the self-intersections of multiplicity $k-1$ in the boundary etc. to produce $z \in I_{r,n-1}$ such that $h(z) = [Q'_k] + h_\ell$–values for $\ell < k+1$.

It is not hard to extend e'' in these constructions since they preserve the ordering of the branches of the immersion e' at the self-intersections. □

Finally we turn to the diagram in theorem 1.14. According to propositions 8.3 and 8.6 we may identify $\Omega^{fr}_{m-n}(\Lambda(S^n, y_0))$ with $I_{m-n,n-1}$ and then apply the selfintersection isomorphism h.

Let us calculate $h \circ \widetilde{\deg}$. Given $[f] \in \pi_m(S^n, y_0)$, pick a regular value $* \in S^n$ near, but different from, the basepoint $y_0 = f(x_0)$. In view of the bijection 8.1, applied to $V = \mathbb{R}^{m-1}$, we may assume that the inclusion

$$(8.7) \qquad e_C \ : \ C = f^{-1}(\{*\}) \quad \subset \ S^m - \{x_0\} \ = \ \mathbb{R}^m$$

of the coincidence manifold projects to a framed immersion e'_C into $\mathbb{R}^{m-1}$. Now let G denote a contraction of e_C in the negative x_m–direction towards $\infty = x_0 \in S^m$. Then $f \circ G$, together with a path θ_* in S^n from $*$ to y_0, describe a map

$$v \ : \ (C \times [0,1], C \times \{0,1\}) \ \longrightarrow \ (S^n, y_0)$$

such that

$$(8.8) \qquad \widetilde{\deg}(f) \ = \ [C, \text{ adjoint of } v]$$

(compare 3.2, 7.1, and 7.2).

In order to determine $h \circ \widetilde{\deg}(f)$, we now apply the discussion of 8.1 to the case $V = C$ and to the map v. Pick a point $*' \in S^n - \{y_o\} = \mathbb{R}^n$ near $*$, but with strictly larger x_n–coordinate and such that $*'$ also avoids the image of the path θ_*. Then the inclusion

$$e_Q \ = \ (e'_Q, e''_Q) \ \ : \ \ Q := v^{-1}(\{*'\}) \quad \subset \quad C \times \mathbb{R}$$

projects to a framed immersion e'_Q of codimension $n-1$; the branches of e'_Q through a point $c \in C$ correspond to those branches of the immersion $e'_C : C \looparrowright \mathbb{R}^{m-1}$ (compare 8.7) which pass *under* the point c w.r. to the x_m–coordinate e''_C. Therefore we can identify $\widetilde{\deg}(f)$ (cf. 8.8), via the isomorphism in proposition 8.3, with the joint bordism class $[e'_Q, Q, C, e''_Q] \in I_{m-n,n-1}$. Moreover the $(k-1)$–tuple point manifold of e'_Q (which represents $h_k(\widetilde{\deg}(f))$, cf. 8.5) equals essentially the k-tuple point locus of the "compression" e'_C of e_C (cf. 8.7). Thus, according to [KS] (see, in particular, p. 287, l. 20, and theorem 3.2) it also represents the stabilized James-Hopf invariant (cf. [J]) – at least up to a sign which depends only on k and which we subsume into the definition of h_k.

In view of proposition 7.5 and of the identity

$$0 \ = \ \widetilde{\omega}(a \circ f, f) \ = \ \widetilde{\deg}(a \circ f) + (-1)^n \operatorname{inv}_*(\widetilde{\deg}(f))$$

the remaining claims in theorem 1.14 follow from

Proposition 8.9. *Let* inv *denote the involution on* $\Lambda(S^n, y_0)$ *which inverses the direction of a loop, and let* inv_* *be the induced involution on* $\Omega_*^{fr}(\Lambda(S^n, y_0))$.
Then for $k \geq 1$

$$h_k \circ \text{inv}_* \quad = \quad -(-1)^{k+(n-1)\binom{k-1}{2}} h_k \quad ;$$

moreover for all $[f] \in \pi_*(S^n)$ *we have*

$$\Gamma_k(f) := E^\infty \circ \gamma_k(f) \quad = \quad -(-1)^{k+(n-1)\binom{k}{2}} \Gamma_k(f)$$

and

$$\Gamma_k(r \circ f) \quad = \quad (-1)^k \Gamma_k(f)$$

where $r : (S^n, y_0) \to (S^n; y_0)$ *is a reflection. (Note that* $a \sim r^{n+1}$*).*

Proof. If in the construction of h_k a given element $\lambda \in \Omega_*^{fr}(\Lambda(S^n, y_0))$ corresponds to a quadruple (e', Q, V, e'') as in 8.2 and 8.3 then $\text{inv}_*(\lambda)$ corresponds to the "reflected" quadruple $(\overline{e}', Q, V, -e'')$. Here e' and $\overline{e}'$ denote the same immersion, but with opposite framing: one of the frame vectors is replaced by its negative; this is needed to make up for the reflection in the $\mathbb{R}$-component which takes e'' to $-e''$. Thus when we compare $h_k(\text{inv}_*(\lambda))$ to $h_k(\lambda)$ each intersection branch at the $(k-1)$-tuple point set Q_{k-1} (cf. 8.4) contributes a factor -1. Moreover the reflection reverses the ordering of the $(n-1)$-codimensional branches; interchanging them yields $(n-1)((k-2)+(k-1)+\cdots+2+1)$ sign changes. All this adds up to the factor $(-1)^{k-1+(n-1)\binom{k-1}{2}}$ claimed in proposition 8.9.

A similar argument shows that the diagram

$$\begin{array}{ccccc}
\pi_m(S^n) & \cong & \pi_{m-1}(\Lambda S^n) & \xrightarrow{\Gamma_k} & \pi^S_{m-1-k(n-1)} \\
\Big\downarrow{\scriptstyle -\,\text{id}} & & \Big\downarrow{\scriptstyle \text{inv}_*} & & \Big\downarrow{\scriptstyle (-1)^{k+(n-1)\binom{k}{2}}\cdot \text{id}} \\
\pi_m(S^n) & \cong & \pi_{m-1}(\Lambda S^n) & \xrightarrow{\Gamma_k} & \pi^S_{m-1-k(n-1)}
\end{array}$$

commutes. Also any given reflection r of S^n can be deformed until it preserves the splitting $S^n - \{\infty\} \cong \mathbb{R}^{n-1} \times \mathbb{R}$ together with the $\mathbb{R}$-values. Thus it preserves also the ordering of the selfintersection branches, but it reverses the normal framing of a projected immersion e' into $V = \mathbb{R}^{m-1}$ (compare 8.2). At the k-tuple point set this amounts to a k-fold change of signs. □

Example 8.10. The image of the homomorphism

$$\widetilde{\deg}_2 \;=\; \Gamma_2 \;:\; \pi_{12}(S^5) \cong \mathbb{Z}_{30} \;\longrightarrow\; \pi_3^S \cong \mathbb{Z}_{24}$$

lies in a group of order 2.

Proof of corollary 1.16. We use the tables in chapter XIV of Toda's book [T]. Also we identify elements of $\pi_m(S^n)$ via the Pontryagin-Thom construction with bordism classes of closed smoothly embedded $(m-n)$-manifolds $Q \subset \mathbb{R}^m$ which are equipped with a framing (i.e. a nonstable trivialisation of the normal bundle

$\nu(Q, \mathbb{R}^m)$). The stable suspension homomorphism E^∞ forgets the embedding and retains only the stably framed manifold Q.

Thus we can represent e.g. the generator η_2 of $\pi_3(S^2) \cong \mathbb{Z}$ by a framed circle in $\mathbb{R}^3$ which projects to the framed figure-8 immersion in $\mathbb{R}^2$, and we see that

$$\Gamma(\eta_2) \;=\; (1,1) \;\in\; \mathbb{Z}_2 \oplus \mathbb{Z} \;\cong\; \pi_1^S \oplus \pi_0^S \;.$$

Similarly $\eta_2 \circ \eta_3 \in \pi_4(S^2)$ and $\eta_2 \circ \eta_3 \circ \eta_4 \in \pi_5(S^2)$ are represented by embedded tori $(S^1)^2$ and $(S^1)^3$ which suspend to the nontrivial elements $\eta^2 \in \pi_2^S$ and $\eta^3 \in \pi_3^S$, resp. It follows that Γ is injective on $\pi_{n+1}(S^n)$, $n \geq 2$, as well as on the groups $\pi_{n+2}(S^n)$, $n \geq 2$, and $\pi_5(S^2)$ which all have order 2.

To finish the proof consider the following exact pieces of EHP-sequences (cf. [W], p. 542)

$$\begin{array}{ccccccccc}
\pi_6(S^3) & \xrightarrow{E} & \pi_7(S^4) & \xrightarrow{H} & \pi_7(S^7) & \longrightarrow & \pi_5(S^3) & \xrightarrow{\cong} & \pi_6(S^4) \\
\cong & & \cong & & \cong & & \cong & & \cong \\
\mathbb{Z}_{12} & & \mathbb{Z} \oplus \mathbb{Z}_{12} & & \mathbb{Z} & & \mathbb{Z}_2 & & \mathbb{Z}_2
\end{array}$$

and

$$\begin{array}{ccccccccc}
\pi_9(S^5) & \xrightarrow{0} & \pi_9(S^9) & \xrightarrow{P} & \pi_7(S^4) & \xrightarrow{E^\infty} & \pi_3^S & \longrightarrow & 0 \\
\cong & & \cong & & \cong & & \cong & & \\
\mathbb{Z}_2 & & \mathbb{Z} & & \mathbb{Z} \oplus \mathbb{Z}_{12} & & \mathbb{Z}_{24} & & .
\end{array}$$

Clearly $H = E^\infty \circ \gamma_2$ is onto here; moreover, E and $E^\infty \oplus H$ are injective. This implies our claim also for $\pi_6(S^3)$ and $\pi_7(S^4)$. □

Proof of corollary 1.17. First recall that for odd n the Whitehead product $[f] := [\iota_n, \iota_n]$ lies in the kernel of the suspension homomorphism E (and hence of E^∞) and also of γ_2 (since $2[\iota_n, \iota_n] = 0$, cf. [W], p. 474 and 485). Moreover, by the famous result of F. Adams on odd Hopf invariants (and again by an EHP-sequence argument) $[\iota_n, \iota_n] \neq 0$ if $n \neq 1, 3, 7$.

In the remaining cases of corollary 1.17 an inspection of Toda's tables (cf. [T], p. 186–188) shows that Γ is also not injective: the domain turns out to be larger than the relevant torsion part of the target group, or the orders of group elements are not compatible; in view of the relation $2\Gamma_4 \equiv 0$ this argument works also for

$$\Gamma \;:\; \pi_{24}(S^6) \;=\; \mathbb{Z}_{24} \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_2 \longrightarrow (\mathbb{Z}_8 \oplus \mathbb{Z}_2) \oplus (\mathbb{Z}_6 \oplus \mathbb{Z}_2) \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \oplus \mathbb{Z}_{24}.$$

□

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