

Selfcoincidences and roots in Nielsen theory

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Abstract. Given two maps f_1 and f_2 from the sphere S^m to an n -manifold N , when are they loose, i.e. when can they be deformed away from one another? We study the geometry of their (generic) coincidence locus and its Nielsen decomposition. On one hand the resulting bordism class of coincidence data and the corresponding Nielsen numbers are strong looseness obstructions. On the other hand the values which these invariants may possibly assume turn out to satisfy severe restrictions, e.g. the Nielsen numbers can only take the values 0, 1 or the cardinality of the fundamental group of N . In order to show this we compare different Nielsen classes in the root case (where f_1 or f_2 is constant) and we use the fact that all but possibly one Nielsen class are inessential in the selfcoincidence case (where $f_1 = f_2$). Also we deduce strong vanishing results.

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1. Introduction

This work is dedicated to Albrecht Dold whose enormous influence on the development of topology continues to this very day. In fact, it was his stimulating paper [DG] (written jointly with Daciberg Gonçalves) which got me interested in coincidences of maps and which inspired me to introduce a related obstruction theory (cf. [K2]–[K7]). Here we apply it to the special case where the domain is a sphere.

Throughout this paper let $f_1, f_2: S^m \rightarrow N$ be two (continuous) maps into the n -dimensional smooth connected manifold N without boundary. We will also assume that $m, n \geq 2$ (the remaining cases being well understood).

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We want to make the coincidence locus

$$(1.1) \quad C(f_1, f_2) = \{x \in M \mid f_1(x) = f_2(x)\}$$

as small as possible (in some sense) by deforming f_1 and f_2 .

Definition 1.2. (compare [DG], p. 293–296 and [GR2], §1, where a different terminology is used for the same concepts; see also [K7], 5.3).

(i) The pair (f_1, f_2) is called *loose* if there exist maps f'_i homotopic to f_i , $i = 1, 2$, whose coincidence locus $C(f'_1, f'_2)$ is empty (i.e. “ f_1 and f_2 can be deformed away from one another”).

(ii) In the special case $f_1 = f_2 =: f$ we have a refined notion: the pair (f, f) is *loose by small deformation* if for every metric on N and for every $\varepsilon > 0$ there exists an ε -approximation \tilde{f} of f such that $\tilde{f}(x) \neq f(x)$ for all $x \in S^m$.

Proposition 1.3. *All pairs of maps $f_1, f_2: S^m \rightarrow N$ are loose if at least one of the following conditions hold:*

- (i) $m < n$; or
- (ii) N is not compact; or
- (iii) the fundamental group $\pi_1(N)$ is not finite; or
- (iv) N is the total space of a Serre fibration with a section and with strictly positive dimensions of the fiber and base (e.g. if N is the product of such spaces).

Proposition 1.4. *Assume that*

(v) N is noncompact or has trivial Euler characteristic $\chi(N)$ (this holds e.g. when n is odd), or

(vi) $\pi_{m-1}(S^{n-1}) = 0$.

Then for all maps $f: S^m \rightarrow N$ the pair (f, f) is loose by small deformation.

Example 1.5 (surfaces). Let $n = 2$ and $m \geq 1$. Then all such pairs (f, f) are loose by small deformation except when $m = 2$ and $N = S^2$ or $\mathbb{R}\mathbb{P}(2)$.

It is not hard to verify the looseness claims in 1.3, 1.4 and 1.5 by elementary considerations (see section 2 below). However, in general a deeper analysis is needed and the following approach has proved to be fruitful.

For every pair of (base point preserving) maps $f_1, f_2: S^m \rightarrow N$ we defined (in [K6]) an invariant

$$(1.6) \quad \omega^\#(f_1, f_2) \in \pi_m(S^m \wedge (\Omega N)^+)$$

which reflects the geometry of a (generic) coincidence submanifold $C \subset S^m$, its normal bundle as well as certain path space data (for details see [K6] or section 3 below). $\omega^\#(f_1, f_2)$ depends only on the (base point preserving) homotopy classes of f_1, f_2 and must vanish if (f_1, f_2) is loose (and, in particular, in all cases listed in proposition 1.3). The converse holds in a “stable” dimension range.

Theorem 1.7. (compare [K4], 1.10 and [K2], 2.2). *Assume $m < 2n - 2$. Then a pair (f_1, f_2) is loose precisely if $\omega^\#(f_1, f_2) = 0$. In the special case $f_1 = f_2 =: f$ the pair (f, f) is loose by small deformation precisely if $\omega^\#(f, f) = 0$.*

Our looseness obstruction is always compatible with additions in homotopy groups:

$$(1.8) \quad \omega^\#(f_1 + f'_1, f_2 + f'_2) = \omega^\#(f_1, f_2) + \omega^\#(f'_1, f'_2)$$

for all $[f_i], [f'_i] \in \pi_m(N)$, $i = 1, 2$.

There is also a canonical involution inv on the group $\pi_m(S^n \wedge (\Omega N)^+)$. It plays a rôle e.g. in the symmetry relation

$$(1.9) \quad \omega^\#(f_2, f_1) = \text{inv}(\omega^\#(f_1, f_2)).$$

Two special settings are very central in coincidence theory.

I. The root case: $f_2 = y_0$ (where the fixed value $y_0 \in N$ is given).

Here our invariant yields the degree homomorphism

$$(1.10) \quad \text{deg}^\# := \omega^\#(-, y_0) : \pi_m(N) \rightarrow \pi_m(S^n \wedge (\Omega N)^+).$$

(For a purely homotopy theoretical interpretation in terms of an enriched Hopf-Ganea invariant see theorem 7.2 in [K6].)

Clearly $\text{deg}^\#$ vanishes on $i_*(\pi_m(N \setminus \{*\}))$ where $i : N \setminus \{*\} \hookrightarrow N$ denotes the inclusion of the complement of a point $*$ in N . It turns out that the resulting sequence

$$(1.11) \quad \pi_m(N \setminus \{*\}) \xrightarrow{i_*} \pi_m(N) \xrightarrow{\text{deg}^\#} \pi_m(S^n \wedge (\Omega N)^+)$$

is very often exact, e.g. when $m \leq 2n - 3$ or $n \leq 2$ or N is a sphere or a (real, complex or quaternionic) projective space of arbitrary dimension (cf. [K6], (6.5)). Thus in these cases $\text{deg}^\#(f)$ is the *complete* looseness obstruction for the pair (f, y_0) .

Question 1.12. Is the sequence (1.11) *always* exact?

The degree $\text{deg}^\#$ can be considered to be our basic coincidence invariant since (in view of (1.8) and (1.9))

$$(1.13) \quad \begin{aligned} \omega^\#(f_1, f_2) &= \omega^\#(f_1, y_0) + \omega^\#(y_0, f_2) \\ &= \text{deg}^\#(f_1) + \text{inv}(\text{deg}^\#(f_2)) \end{aligned}$$

for all $[f_1], [f_2] \in \pi_m(N, y_0)$.

II. The selfcoincidence case: $f_1 = f_2 =: f$.

Consider the bundle $ST(N)$ of unit tangent vectors over N (with respect to some metric) and the resulting exact (horizontal) homotopy sequence as well as the Freudenthal suspension homomorphism E :

$$(1.14) \quad \begin{array}{ccccccc} \cdots & \longrightarrow & \pi_m(ST(N)) & \longrightarrow & \pi_m(N) & \xrightarrow{\partial} & \pi_{m-1}(S^{n-1}) & \longrightarrow & \cdots \\ & & & & & & \downarrow E & & \\ & & & & & & \pi_m(S^n) & & \cdot \end{array}$$

Theorem 1.15. *Given $[f] \in \pi_m(N)$, we have the following logical implications:*

- (i) $\partial([f]) \in \pi_{m-1}(S^{n-1})$ vanishes;
- \Downarrow
- (ii) (f, f) is loose by small deformation;
- \Downarrow (\Downarrow if $N = \mathbb{R}\mathbb{P}(n)$)
- (iii) (f, f) is loose (by any deformation);
- \Downarrow (\Downarrow if $N = S^n$)
- (iv) $\omega^\#(f, f) = 0$;
- \Downarrow
- (v) $E(\partial([f])) = 0$.

Thus $\omega^\#(f, f)$ is just “one desuspension short” of being the *complete* looseness obstruction.

The equivalence of (i) and (ii) was already noted by Dold and Gonçalves in [DG].

Observe also that all conditions (i)–(v) in 1.15 *except* (iii) are compatible with covering projections $p: \tilde{N} \rightarrow N$ (compare [K7], 1.22).

Corollary 1.16. *The conditions (i), \dots , (v) in 1.15 are all equivalent if the suspension homomorphism E , when restricted to $\partial(\pi_m(N))$ (cf. (1.14)), is injective and, in particular, if $m \leq n+3$ or if $m = n+4 \neq 10$ or in the stable dimensional range $m \leq 2n-3$.*

Indeed, in these three dimension settings E is injective whenever $n \equiv 0(2)$. Clearly conditions (i)–(v) in 1.15 are automatically satisfied under the assumptions of proposition 1.4.

Example 1.17 ($N = \mathbb{R}\mathbb{P}(n)$ or S^n). If $m \leq n+4$, then the five conditions in 1.15 are equivalent for all maps $f: S^m \rightarrow \mathbb{R}\mathbb{P}(n)$ and $\tilde{f}: S^m \rightarrow S^n$ (even in the exceptional case $m = n+4 = 10$ since $\pi_{10}(S^6) = 0$).

However, this is no longer true for $m = n+5 = 11$. Indeed, according to [T] and [P] we have in (1.14)

$$\frac{1}{2}H: \pi_{11}(S^6) \xrightarrow{\cong} \mathbb{Z}; \quad \pi_{10}(S^5) \cong \mathbb{Z}_2; \quad \pi_{10}(V_{7,2}) = 0$$

where H denotes the Hopf invariant. Thus ∂ is onto, but E and hence $E \circ \partial$ is trivial here. Therefore, given any map $f: S^{11} \rightarrow \mathbb{R}\mathbb{P}(6)$ and a lifting $\tilde{f}: S^{11} \rightarrow S^6$ of it, the invariants $\omega^\#(f, f)$ and $\omega^\#(\tilde{f}, \tilde{f})$ vanish and the pair (\tilde{f}, \tilde{f}) is loose (see [K6], 1.12; compare also [GW]). But (f, f) is loose and, equivalently, (\tilde{f}, \tilde{f}) is loose by small deformation only (and precisely) if $[f] \in 2 \cdot \pi_{11}(\mathbb{R}\mathbb{P}(6))$ or, equivalently, if $H(\tilde{f}) \equiv 0(4)$ (compare [GR1] and [K7], 1.19; the delicate difference between conditions (ii) and (iii) in 1.15 is further illustrated e.g. in [GR2] and [K7], 1.22).

For simple examples of nontrivial $\omega^\#$ -values consider the case $m = n$. If a map $\tilde{f}: S^n \rightarrow S^n$ has (standard) mapping degree d , then

$$\pm\omega^\#(\tilde{f}, \tilde{f}) = E \circ \partial(\tilde{f}) = (1 + (-1)^n) d$$

in $\pi_n(S^n \wedge (\Omega S^n)^+) \cong \pi_n(S^n) \cong \mathbb{Z}$. Assume that n is even. Then (\tilde{f}, \tilde{f}) is loose if and only if \tilde{f} is nullhomotopic; the same holds for maps from S^n into $\mathbb{R}\mathbb{P}(n)$. This shows that the exception made in example 1.5 is indeed necessary. \square

In previous work (cf. [K6]) we simplified our $\omega^\#$ -invariant by *assuming* that f_1 or f_2 , say f_2 , is “not coincidence producing” (cf. [BS]), i.e.

$$(1.18) \quad (\bar{f}_2, f_2) \text{ is loose for some map } \bar{f}_2: S^m \rightarrow N.$$

Then (f_1, f_2) and $(f_1 - \bar{f}_2, (f_2 - f_2) \sim y_0)$ have a similar coincidence behaviour and

$$(1.19) \quad \omega^\#(f_1, f_2) = \text{deg}^\#(f_1 - \bar{f}_2).$$

In this paper we start from the decomposition (up to homotopy)

$$(f_1, f_2) \sim (f_1 - f_2, y_0) + (f_2, f_2)$$

which is always available without any assumption. In view of (1.8) this implies the basic equation

$$(1.20) \quad \omega^\#(f_1, f_2) = \text{deg}^\#(f_1 - f_2) + \omega^\#(f_2, f_2).$$

Vanishing results concerning the second (“selfcoincidence”) summand are not only interesting in view of theorem 1.15, but they also allow us to reduce our analysis to studying the degree homomorphism $\text{deg}^\#$ or, equivalently, Hopf-Ganea invariants (cf. [K6], 7.2).

It turns out that the cardinality $\#\pi_1(N) \in \mathbb{Z} \cup \{\infty\}$ of the fundamental group of N plays a crucial rôle.

Theorem 1.21. *Assume that*

- (i) $\#\pi_1(N) > 2$ and N is orientable or not; or
- (ii) $\#\pi_1(N) \geq 2$ and N is orientable.

Then $\omega^\#(f, f) = 0$ for all $f: S^m \rightarrow N$.

In particular

$$\omega^\#(f_1, f_2) = \text{deg}^\#(f_1 - f_2)$$

for all $[f_1], [f_2] \in \pi_m(N)$.

Also the set of all possible values of our $\omega^\#$ -invariant is restricted by the relation

$$\{\omega^\#(f_1, f_2)\} = \text{im}(\text{deg}^\#) \subset \ker(\text{id} + \text{inv})$$

where $\text{id} = \text{identity}$ and inv denotes the canonical involution of $\pi_m(S^n \wedge (\Omega N)^+)$ (cf. (1.9) and (1.13)).

If in addition $m < 2n - 2$ or $m \leq n + 3$, then the pair (f, f) is loose by small deformation for every map $f: S^m \rightarrow N$.

On one hand, the proof (given in section 4 below) compares the contributions of the pathcomponents of the loop space ΩN to $\omega^\#(f, f)$; on the other hand, it uses the fact that at most one such pathcomponent can contribute nontrivially to our selfcoincidence invariant.

Example 1.22. As an application consider the case where $m = n$ and N is an orientable n -manifold with $\pi_1(N) = \mathbb{Z}_2$. Then

$$\text{inv} = (-1)^n \text{id} \quad \text{on} \quad \pi_n(S^n \wedge (\Omega N)^+) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

If n is even then $\ker(\text{id} + \text{inv}) = 0$; therefore each pair of maps $f_1, f_2: S^m \rightarrow N$ is loose (since $\omega^\#(f_1, f_2) = 0$).

Example 1.23 (real Grassmann manifolds). Consider the manifold $G_{r,k}$ (or $\tilde{G}_{r,k}$, resp.) of nonoriented (or oriented, resp.) k -planes in \mathbb{R}^r , $0 < k < r$. If r is even, then $G_{r,k}$ is orientable, $\pi_1(G_{r,k}) \neq \{0\}$ and hence $\omega^\#(f, f) = 0$ for all maps from S^m , $m \geq 1$, to $G_{r,k}$ or $\tilde{G}_{r,k}$. This does not seem to follow from 1.3 or 1.4; indeed, the Euler characteristic of $G_{r,k}$ is strictly positive if both r and k are even (see 4.3 below; compare also our example 2.2).

A further vanishing result (quoted from [K6], 1.9; compare also [K7], 5.4) is of interest here.

Proposition 1.24. *Assume $\pi_1(N) \neq 0$. Given $[f] \in \pi_m(N)$, $m \geq 2$, if $j_* \circ \partial([f])$ vanishes then so does $\omega^\#(f, f)$.*

Here $j: S^{n-1} \hookrightarrow N \setminus \{*\}$ denotes the inclusion of the boundary sphere of a small n -ball in N centered at a point $* \in N$. According to [K7], 5.4, the condition $j_* \circ \partial([f]) = 0$ just means that f is not coincidence producing (cf. (1.18)). \square

Let us summarize: *although our selfcoincidence invariant $\omega^\#(f, f)$ is “only one desuspension short” of being a complete looseness obstruction, it vanishes for a great number of maps f defined on spheres* (in contrast, in [K2] $\omega^\#(f, f)$ was shown to be highly nontrivial in many examples where the domain of f is a general closed manifold M).

Corollary 1.25. *Assume $\pi_1(N) \neq 0$ and $n \geq 1$. If $\omega^\#(f, f) \neq 0$ for a map $f: S^m \rightarrow N$ then the following restrictions must all be satisfied:*

- a) n is even and $m \geq n \geq 4$, or else $m = 2$ and $N = \mathbb{RP}(2)$; and
- b) N is closed and nonorientable, $\pi_1(N) = \mathbb{Z}_2$, $\chi(N) \neq 0$; moreover the homomorphism $i_*: \pi_m(N \setminus \{*\}, y_0) \rightarrow \pi_m(N, y_0)$ (induced by the inclusion of N , punctured at some point $* \neq y_0$) is not onto; furthermore, N does not allow a free smooth action by a nontrivial finite group; and
- c) $E \circ \partial([f]) \neq 0$ and $j_* \circ \partial([f]) \neq 0$ (compare (1.14) and 1.21).

Thus an obvious place where to look for nontrivial values of $\omega^\#(f, f)$ are even-dimensional real projective spaces (cf. already example 1.17).

Example 1.26. If $n = 4, 8, 12, 14, 16$ or 20 , then there exist infinitely many homotopy classes $[f] \in \pi_{2n-1}(\mathbb{R}\mathbb{P}(n))$ such that $\omega^\#(f, f) \neq 0$.

We see this with the help of the weaker invariant $\omega(f, f) \in \pi_{m-n}^S$ (cf. (3.15) below and [K2], 2.1), which stabilizes and simplifies $\omega^\#(f, f)$ and is more easily computable.

However, for many maps f, f_1, f_2 from S^m into a nonorientable n -manifold N this approach does not yield nontrivial ω -values. Indeed, $\omega(f, f) = 0$ whenever $2 \cdot \pi_{m-n}^S = 0$, e.g. when $m - n = 1, 2, 4, 5, 6, 8, 9, 12, 14, 16$ or 17 (cf. [T]). Moreover, for all m and n

$$(1.27) \quad (\chi(N) - 1) \omega(f_1, f_2) = 0$$

where $\chi(N)$ stands for the Euler characteristic of N .

Example 1.28 ($\mathbf{N} = \mathbf{G}_{5,2}$; compare 1.23). The invariant $\omega(f_1, f_2)$ vanishes for all maps $f_1, f_2: S^m \rightarrow G_{5,2}$, $m \geq 1$. In particular, the induced homomorphism

$$\text{coll}_* : \pi_m(G_{5,2}) \rightarrow \pi_m(S^6)$$

is trivial for $m \leq 10$ where coll denotes the degree one map which collapses all but an open topdimensional ball into a point. (Note that e.g. for $m = 6, 7$ or 8 , resp., $\pi_m(G_{5,2}) \cong \pi_m(V_{5,2})$ is isomorphic to $\mathbb{Z}_2, \mathbb{Z} \oplus \mathbb{Z}_2$ and \mathbb{Z}_2 , resp.; cf. [P]). \square

In section 4 below we will deduce the vanishing theorem 1.21 and relations such as (1.27) and 1.28 from a careful analysis of the root case in section 3.

While our looseness obstructions lie in complicated groups which are usually hard to compute, they give rise to simple numerical invariants (defined in section 5 below). These generalize the Nielsen numbers which have played such a central rôle in topological fixed point theory (cf. e.g. [B] and [K6], 1.12 (iv)). In analogy to the classical procedure, our Nielsen numbers $N^\#(f_1, f_2) \geq N(f_1, f_2)$ count those ‘‘Reidemeister classes’’ $A \in \pi_0(\Omega N) = \pi_1(N)$ which make an essential contribution to $\omega^\#(f_1, f_2)$ and $\tilde{\omega}(f_1, f_2)$, resp.

Theorem 1.29. (cf. [K6], 1.2). *For every pair (f'_1, f'_2) of maps homotopic to (f_1, f_2) the number of pathcomponents of the coincidence subspace $C(f'_1, f'_2) \subset S^m$ is at least $N^\#(f_1, f_2)$.*

There are quite a few sample settings where $N^\#(f_1, f_2)$ is the best possible such lower bound (see e.g. the ‘‘Wecken theorems’’ in [K6], 1.3, 1.12, 1.13, 1.14, ...).

Theorem 1.30. *For all maps $f_1, f_2: S^m \rightarrow N$ we have:*

$$\begin{aligned} N^\#(f_1, f_2) = 0 & \quad \text{if and only if} \quad \omega^\#(f_1, f_2) = 0 ; \\ N(f_1, f_2) = 0 & \quad \text{if and only if} \quad \tilde{\omega}(f_1, f_2) = 0. \end{aligned}$$

We call this the norm property of our Nielsen numbers: they decide whether elements in the image group of our ω -invariants are zero, just like norms of vectors do. (Recall, in addition, analogues of the triangular inequality, cf. [K6], 6.2.)

By definition $N^\#(f_1, f_2)$ is among the integers between 0 and $\#\pi_1(N)$. But few of these occur actually as Nielsen numbers.

Theorem 1.31. *Let $k \in \mathbb{Z} \cup \{\infty\}$ be the number of elements in $\pi_1(N)$. Then for each pair of maps $f_1, f_2: S^m \rightarrow N$ the Nielsen numbers $N^\#(f_1, f_2)$ and $N(f_1, f_2)$ may assume only the two values 0 or k or, if $k = 2$ and N is an even-dimensional, closed, nonorientable manifold with nontrivial Euler characteristic, also 1 as a third possible value.*

Here again (as also already in [K7]) the case $N = \mathbb{RP}(n)$ turns out to be particularly interesting. E.g. assume that $m = n$ is even and let $p: S^n \rightarrow \mathbb{RP}(n)$ be the canonical covering map. Then $N^\#(f_1, f_2) = N(f_1, f_2)$ equals 0, 1 and 2, resp., when $(f_1, f_2) = (y_0, y_0)$, (p, p) and (p, y_0) , resp. (cf. the end of section 4 below).

Remark 1.32. It will be convenient to consider base point preserving maps and homotopies in sections 3–5 below. Recall, however, that looseness and Nielsen numbers depend only on free homotopy classes and so does the vanishing of our ω -invariants (see e.g. [K6], 1.2, 2.1 and the appendix there).

Remark 1.33. Our approach can also be extended to the setting of fibre preserving maps between smooth fibrations. The resulting coincidence invariants are closely related to A. Dold’s fixed point index which was defined and studied in [D].

Conventions and Notations. Throughout this paper N denotes a smooth connected manifold without boundary (Hausdorff and having a countable basis). Our notation will often not distinguish between a constant map and its value.

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2. Looseness

In this section we use rather elementary techniques to establish the looseness results in propositions 1.3 and 1.4 as well as example 1.5.

Lemma 2.1. *Let $y_1 \neq y_2$ be different points in N and assume that the homomorphism $i_*: \pi_m(N \setminus \{y_1\}, y_2) \rightarrow \pi_m(N, y_2)$ induced by the inclusion map is onto. Then for all maps $f_1, f_2: S^m \rightarrow N$ the pair (f_1, f_2) is loose.*

Proof. Let S_+^m and S_-^m denote the halfspheres defined by $x_1 \geq 0$ and $x_1 \leq 0$, resp., $x = (x_1, \dots, x_{m+1}) \in S^m$. Then f_2 is homotopic to a map f'_2 such that $f'_2(S_+^m) \subset N \setminus \{y_1\}$ and $f'_2|_{S_-^m} \equiv y_2$. Similarly, $f_1 \sim f'_1$ such that $f'_1|_{S_+^m} \equiv y_1$ and $f'_1(S_-^m) \subset N \setminus \{y_2\}$. Clearly the pair (f'_1, f'_2) is coincidence free. \square

Proposition 1.3 follows as a corollary since each of the conditions (i) through (iv) imply that i_* is onto. If $m < n$, this is seen by making a map $f: S^m \rightarrow N$ transverse to $\{y_1\}$. If N is not compact, deform f by an isotopy along a smoothly embedded path which starts in $N \setminus f(S^m)$, ends in y_1 and avoids y_2 .

If the universal covering space $p: \tilde{N} \rightarrow N$ has infinitely many layers, consider a lifting $\tilde{f}: S^m \rightarrow \tilde{N}$ of f . Apply an isotopy along suitable disjoint paths in \tilde{N} which start in $\tilde{N} \setminus \tilde{f}(S^m)$ and end in the finitely many points of $\tilde{f}(S^m) \cap p^{-1}(\{y_1\})$. The corresponding homotopy in N deforms f into $N \setminus \{y_1\}$.

Finally, in case (iv) of proposition 1.3 the exact homotopy sequence of the fibration splits. Thus f can be deformed into the union of the fibre and the image of the section. \square

Example 2.2. For $r = 2r' > 2$ and $m \geq 1$ every pair of maps $f_1, f_2: S^m \rightarrow G_{r,2}$ (into the Grassmann manifold of 2-planes in \mathbb{R}^r) is loose.

Indeed, due to the complex structure on $\mathbb{R}^r = \mathbb{C}^{r'}$, the fibration $S^{r-2} \hookrightarrow V_{r,2} \rightarrow S^{r-1}$ (of the Stiefel manifold of 2-frames in \mathbb{R}^r) has a canonical section; hence

$$\pi_m(V_{r,2}) \cong \pi_m(S^{r-2}) \oplus \pi_m(S^{r-1}).$$

For $m \geq 3$ this group is also isomorphic to $\pi_m(G_{r,2})$, and the summands correspond to the subspaces

$$A := \{\text{2-planes containing the base vector } e_r\}$$

and

$$B := \{\text{complex lines in } \mathbb{C}^{r'}\}.$$

Since $A \cup B \subsetneq G_{r,2}$, our claim follows from lemma 2.1. \square

Proof of proposition 1.4. A homotopy lifting argument shows that (f, f) is loose by small deformation if and only if the pulled back bundle $f^*(TN)$ has a nowhere vanishing section over S^m (compare [DG]). If N is noncompact or $\chi(N) = 0$, then the tangent bundle TN itself has a nonzero section over $f(S^m)$ which we can pull back. In any case every n -plane bundle over S^m allows a section with a single zero; it can be removed if its “index map” $q: S^{m-1} \rightarrow S^{n-1}$ (compare [Wy] or also [K6], (28)) is nullhomotopic. \square

The claim in example 1.5 follows since $\pi_{m-1}(S^1) = 0$ except when $m = 2$. Furthermore $\pi_2(N) = 0$ for every surface other than $N = S^2$ or $\mathbb{RP}(2)$.

3. Looseness obstructions

In this section we recall the definitions of the various versions of the ω -invariants. (For more details we refer to [K6]; see also [K4] and [K2].) Moreover we compare the Nielsen components of $\omega^\#$ in the root case (in 3.12). This will be needed to establish our main results.

Unless specified otherwise we will assume that $m, n \geq 2$. Fix basepoints $x_0 \in S^m$ and $y_0 \in N$, and a local orientation of N at y_0 . Throughout the remainder of this paper we consider (continuous) maps $f_1, f_2, f, \dots: (S^m, x_0) \rightarrow (N, y_0)$. Our notation will not distinguish between a constant map and its value.

After a suitable approximation we may assume that $(f_1, f_2): S^m \rightarrow N \times N$ is smooth and transverse to the diagonal

$$\Delta = \{(y, y) \in N \times N \mid y \in N\}.$$

Then the coincidence set

$$(3.1) \quad C = C(f_1, f_2) = (f_1, f_2)^{-1}(\Delta) = \{x \in S^m \mid f_1(x) = f_2(x)\}$$

is a smooth closed n -codimensional submanifold of S^m . The map (f_1, f_2) induces an isomorphism of (normal) vector bundles

$$(3.2) \quad \nu(C, S^m) \cong (f_1, f_2)^*(\nu(\Delta, N \times N)) \cong f_1^*(TN)|_C.$$

Now pick a homotopy $G: C \times I \rightarrow S^m$ which deforms the inclusion map $g: C \hookrightarrow S^m$ to the constant map with value x_0 . (Such a contraction exists and is unique up to homotopy rel $(0, 1)$ since $n \geq 2$ and the complement of any point in S^m allows “linear” homotopies.) Then G induces a vector bundle isomorphism from $f_1^*(TN)|_C = g^*(f_1^*(TN))$ to $C \times T_{y_0}(N)$. Composing this with (3.2) and using our choice of a local orientation of N at y_0 , we get a framing

$$(3.3) \quad \bar{g}^\#: \nu(C, S^m) \xrightarrow{\cong} C \times \mathbb{R}^n.$$

Furthermore we obtain the map

$$(3.4) \quad \tilde{g}: C \rightarrow \Omega(N, y_0) =: \Omega N$$

which assigns the (concatenated) loop

$$(3.5) \quad y_0 = f_1(x_0) \xrightarrow{(f_1 \circ G(x, -))^{-1}} f_1(x) = f_2(x) \xrightarrow{f_2 \circ G(x, -)} f_2(x_0) = y_0$$

to $x \in C$.

The resulting bordism class

$$(3.6) \quad \omega^\#(f_1, f_2) = [C, \bar{g}^\#, \tilde{g}]$$

of the framed compact submanifold $C \subset S^m$ (cf. (3.1); (3.3)) together with the map \tilde{g} (cf. (3.4)) depends only on the homotopy classes $[f_i] \in \pi_m(N, y_0)$, $i = 1, 2$. Via the Pontryagin-Thom procedure, $\omega^\#(f_1, f_2)$ can also be interpreted as an element in the m -th homotopy group of the Thom space $S^n \wedge (\Omega N)^+$ of the trivial n -plane bundle over the loop space ΩN . (Here “+” stands for a disjointly added point. Note that the bordism theories of submanifolds in S^n and \mathbb{R}^n are equivalent in codimensions $n \geq 2$; thus it is not necessary that $f_1(x_0) \neq f_2(x_0)$, as was assumed in [K6].)

If we ignore the map \tilde{g} we obtain the simpler invariant

$$(3.7) \quad \underline{\omega}^\#(f_1, f_2) = [C, \bar{g}^\#] \in \pi_m(S^n).$$

However, often this means a considerable loss of information. Indeed, in general the loop space ΩN has a very rich topology and, in particular, can be highly disconnected. Already its decomposition into pathcomponents leads to the important “Nielsen decomposition” of coincidence sets, as follows.

Given a pathcomponent $A \in \pi_0(\Omega N) = \pi_1(N, y_0)$, restrict your attention to the corresponding partial coincidence manifold

$$C_A := C_A(f_1, f_2) := \tilde{g}^{-1}(A) \subset C(f_1, f_2)$$

which is again a closed n -codimensional submanifold of S^n and endowed with the restricted framing $\bar{g}_A^\# = \bar{g}^\#|$ and the map $\tilde{g}_A = \tilde{g}|: C_A \rightarrow A \subset \Omega N$. This leads to the invariants

$$(3.8) \quad \omega_A^\#(f_1, f_2) = [C_A, \bar{g}_A^\#, \tilde{g}_A] \in \pi_m(S^n \wedge A^+)$$

and

$$(3.9) \quad \underline{\omega}_A^\#(f_1, f_2) = [C_A, \bar{g}_A^\#] \in \pi_m(S^n).$$

Example 3.10. For all $[f] \in \pi_m(N, y_0)$ we have $\underline{\omega}^\#(y_0, f) = \text{coll}_*([f])$. Here

$$\text{coll}: N \rightarrow N/(N \setminus \overset{\circ}{B}^n) = B^n/\partial B^n = S^n$$

is the map which collapses the complement of a small ball B^n near $y_0 \in \partial B^n$ to a point and preserves the local orientation of N . \square

In the root case we will now compare the various Nielsen components of $\omega^\#$.

Note there are two canonical isomorphisms

$$(3.11) \quad \varrho_{A*}, \lambda_{A*}: \pi_m(S^n \wedge A_0^+) \xrightarrow{\cong} \pi_m(S^n \wedge A^+)$$

(compare (3.8)) where A_0 denotes the pathcomponent of ΩN containing the constant loop. They are induced by the homotopy equivalences $\varrho_A, \lambda_A: A_0 \rightarrow A$ which compose each loop $\ell \in A_0$ to the right (or left, resp.) with a fixed loop $\ell_A \in A$; e.g. $\varrho_A(\ell)$ travels first along ℓ and then along ℓ_A .

To simplify notations we will write $\omega_0^\#$ for $\omega_{A_0}^\#$.

Proposition 3.12. *Given a map $f: (S^m, x_0) \rightarrow (N, y_0)$ and $A \in \pi_1(N)$, we have*

$$\omega_A^\#(f, y_0) = \varrho_{A*} \left(\omega_0^\#(f, y_0) \right) \quad \text{and} \quad \underline{\omega}_A^\#(f, y_0) = \underline{\omega}_0^\#(f, y_0)$$

as well as

$$\omega_A^\#(y_0, f) = \iota_{A*} \circ \lambda_{A*} \left(\omega_0^\#(f, y_0) \right) \quad \text{and} \quad \underline{\omega}_A^\#(y_0, f) = \iota_{A*} \left(\underline{\omega}_0^\#(y_0, f) \right)$$

where ι_{A*} composes the framing with an orientation preserving (or reversing) automorphism of \mathbb{R}^n according as A is (or not) orientation preserving, i.e. the tangent bundle of N , when pulled by $\ell_A: S^1 \rightarrow N$, $[\ell_A] = A$, becomes trivial or, equivalently, A lies in the kernel of the composed homomorphism

$$w_1(N): \pi_1(N, y_0) \rightarrow H_1(N) \rightarrow \mathbb{Z}_2$$

which evaluates the first Stiefel-Whitney class of N .

Proof (for some of the following arguments compare also the proof of theorem 4.3 in [K6]).

In view of proposition 1.3 (iii) we need to consider only the case when $\pi_1(N)$ is finite. Thus the fibre $p^{-1}(\{y_0\})$ in the universal covering space $p: \tilde{N} \rightarrow N$ consists of points $\{\tilde{y}_1, \dots, \tilde{y}_k\}$ which all lie in a suitable embedded n -ball V in \tilde{N} . Now lift f to a map $\tilde{f}: S^m \rightarrow \tilde{N}$. After a suitable homotopy we may assume that \tilde{f} is smooth, with regular value \tilde{y}_1 , and agrees on a tubular neighbourhood $U \cong \tilde{C} \times V \subset S^m$ of $\tilde{C} := \tilde{f}^{-1}(\{\tilde{y}_1\})$ with the projection to $V \subset \tilde{N}$. Then

$$C(f, y_0) = C(y_0, f) = f^{-1}(\{y_0\}) = \tilde{f}^{-1}(\{\tilde{y}_1, \dots, \tilde{y}_k\})$$

consists of the “parallel” copies

$$\tilde{C}_i := \tilde{C} \times \{\tilde{y}_i\} \subset U, \quad i = 1, \dots, k.$$

Note that the straight path γ_{ij} joining \tilde{y}_i to \tilde{y}_j in the ball V yields an isotopy which moves \tilde{C}_i to \tilde{C}_j in $U \subset S^m$, $1 \leq i, j \leq k$. We can compose it with a contraction $G|_{\tilde{C}_j \times I}$ of \tilde{C}_j in order to get the required contraction of \tilde{C}_i in S^m . This extra part of the contraction induces a concatenation of the path in (3.5) with the loop $p \circ \gamma_{ij}^{-1}$. On the other hand the isotopy is compatible with the framings of \tilde{C}_i and \tilde{C}_j if $p \circ \gamma_{ij}$ preserves the local orientation of N or in case we are dealing with $\omega^\#(f, y_0)$ since then

$$\nu(\tilde{C}_t, S^n) \cong \tilde{C}_t \times TV \cong f^*(TN)|_{\tilde{C}_t},$$

$0 \leq t \leq 1$, throughout the isotopy. Thus let us consider $\omega^\#(y_0, f)$. Here \tilde{C}_i and \tilde{C}_j are framed via the orientations of $T_{\tilde{y}_i}\tilde{N}$ and $T_{\tilde{y}_j}\tilde{N}$, resp., induced by p from the given orientation of N at y_0 . If $p \circ \gamma_{ij}$ reverses it, then the two framings correspond to opposite orientations of V , and the isotopy does not change this. \square

Next let us recall how the involution inv in (1.9) is defined (cf. [K6], p. 632). Given $[C, \bar{g}^\#, \tilde{g}] \in \pi_m(S^n \wedge (\Omega N)^+)$, inv retains the submanifold $C \subset S^m$ but changes its framing by $(-1)^n \cdot \alpha$ where the vector bundle automorphism α of the trivial bundle $C \times \mathbb{R}^n$ is determined by TN and the homotopy $C \times I \rightarrow N$ which evaluates \tilde{g} (cf. [K4], 3.1). In addition, we have to compose \tilde{g} with the selfmap of ΩN which inverts the direction of loops.

Example 3.13 ($\mathbf{m} = \mathbf{n} \equiv \mathbf{0(2)}$, N orientable). Here the framings (=coorientations) of the (isolated) points $x \in C$ remain unchanged, but we have to travel backwards along each loop $\tilde{g}(x)$.

Warning. In general the symmetry relation (1.9) does not necessarily extend to $\underline{\omega}^\#$: the weaker invariant $\underline{\omega}^\#(f_2, f_1)$ may depend on more than just its weak counterpart $\underline{\omega}^\#(f_1, f_2)$.

A weaker version of our looseness obstruction $\omega^\#(f_1, f_2)$ is often much easier to handle and to compare. If we forget about *embeddings* of coincidence manifolds

into S^m and if we keep track only of stabilized normal bundles we obtain the invariants $\tilde{\omega}_A(f_1, f_2)$, $A \in \pi_1(N)$, and

$$(3.14) \quad \tilde{\omega}(f_1, f_2) \in \Omega_{m-n}^{\text{fr}}(\Omega N) \cong \pi_{m+q}(S^{m+q} \wedge (\Omega N)^+), \quad q \gg 0,$$

which lie in the $(m-n)$ -dimensional framed bordism group of the loop space ΩN of N (cf. [K4]). Similarly

$$(3.15) \quad \omega(f_1, f_2) := \underline{\omega}(f_1, f_2), \quad \underline{\omega}_A(f_1, f_2) \in \Omega_{m-n}^{\text{fr}} = \pi_{m-n}^S$$

(compare [K2], 1.4) are the stabilized versions of $\omega^\#(f_1, f_2)$ and $\omega_A^\#(f_1, f_2)$.

In general this stabilization procedure leads to a loss of information; not so, however, in the dimension range $m < 2n - 2$ where $\tilde{\omega}(f_1, f_2)$ is just as strong as $\omega^\#(f_1, f_2)$ and, in fact, is the *complete* looseness obstruction for the pair (f_1, f_2) (cf. [K6], (16) and [K4], theorem 1.10).

4. Selfcoincidences

In this section we prove theorem 1.21 and further vanishing results and discuss their consequences.

Given any map $f: (S^m, x_0) \rightarrow (N, y_0)$ let us look at the coincidence data of the pair (f, f) . Clearly the map \tilde{g} (cf. (3.4)) is canonically nullhomotopic since $f_1 = f_2 = f$ in (3.5). For small generic approximations of (f, f) the partial coincidence manifolds C_A are empty (and hence nullbordant) whenever $A \in \pi_1(N)$ is nontrivial.

We conclude that

$$(4.1) \quad \omega^\#(f, f) = s_*(\omega_0^\#(f, f))$$

where

$$s_*: \pi_m(S^n) \rightarrow \pi_m(S^n \wedge (\Omega N)^+)$$

is induced by the inclusion of the constant loop into ΩN (recall also the notation of 3.12).

Proof of theorem 1.21. We base our argument on the identity

$$(4.2) \quad \omega_A^\#(f, f) = \omega_A^\#(f, y_0) + \omega_A^\#(y_0, f)$$

(valid for all $A \in \pi_1(N)$; cf. (1.13)) and on proposition 3.12. The assumptions in 1.21 just mean that the homomorphism $w_1: \pi_1(N) \rightarrow \mathbb{Z}_2$ (cf. 3.12) is not injective. In other words, there exists an element A of $\pi_1(N)$ which is both nontrivial and orientation preserving. Then on one hand $\omega_A^\#(f, f) = 0$. On the other hand 3.12 and (4.2) combine to show that $\omega_A^\#(f, f) = \omega_0^\#(f, f)$. Thus in view of (4.1) the full $\omega^\#$ -invariant of the pair (f, f) – and of every pair freely homotopic to it (cf. remark 1.32) – vanishes.

To complete the proof of theorem 1.21 note that

$$\omega^\#(f, f) = \omega^\#(f, y_0) + \omega^\#(y_0, f) = (\text{id} + \text{inv})(\text{deg}^\#(f))$$

(cf. (1.8), (1.9) and (1.10)). Recall also corollary 1.16. \square

Corollary 1.25 follows now from 1.3, 1.4, (1.11), (1.13), 1.15 (to be discussed below), 1.21, 1.24 and the compatibility of $\underline{\omega}(f, f)$ with covering maps (compare (4.1)).

As for the conclusion of example 1.22 see also 3.13. Similarly, the statement in example 1.23 is a consequence of theorem 1.21 and the following wellknown

Fact 4.3 (real Grassmann manifolds). *Given integers $0 < k < r$, the manifold $G_{r,k}$ of all k -dimensional linear subspaces of \mathbb{R}^r enjoys the following properties:*

1. $\pi_1(G_{r,k}) \cong \mathbb{Z}_2$ if $r > 2$;
2. $G_{r,k}$ is orientable if and only if r is even;
3. $\dim(G_{r,k}) = k(r - k)$;
4. the Euler characteristic vanishes if $k \not\equiv r \equiv 0(2)$ and equals the binomial coefficient

$$\chi(G_{r,k}) = \binom{\lfloor r/2 \rfloor}{\lfloor k/2 \rfloor} > 0$$

in all other cases.

Question 4.4. What about $\omega^\#(f, f)$ for maps into $G_{r,k}$ or $\tilde{G}_{r,k}$ when r is odd and $k > 2$?

Proof of theorem 1.15. A homotopy lifting argument shows that (f, f) is loose by small deformation precisely if the pulled back vector bundle $f^*(TN)$ has a nowhere vanishing section over S^m (cf. [DG] or [K7], 5.3) or, equivalently, if f lifts to $ST(N)$ (cf. (1.14)). Thus (i) \iff (ii).

Recall also from [K7], 5.7 that $\pm E(\partial[f])$ equals the invariant $\underline{\omega}^\#(f, f)$ which is just as strong as $\omega^\#(f, f)$ (cf. (4.1)).

Next compare the fibre homotopy sequence of $ST(N)$ and of the configuration space $\tilde{C}_2(N) = N \times N \setminus \Delta$ (cf. [K7], 5.4). We see that (iii) \implies (i) provided the induced homomorphism j_* (cf. 1.24) is injective. This is the case e.g. when $N = \mathbb{R}\mathbb{P}(n)$.

Finally recall that (iv) implies (iii) when $N = S^n$; this is a special case of [K6], 1.12. \square

Next we prove corollary 1.16. Clearly E (cf. (1.14)) is injective if $m = n$ or $m \leq 2n - 3$ or $n = 2$ (since $\pi_{m-1}(S^1) = 0$ for $m > 2$). Moreover $\partial(\pi_m(N)) = 0$ if n is odd since then the fibration $ST(N) \rightarrow N$ allows a section over each compactum (compare (1.14)). Thus our corollary follows from

Proposition 4.5. *Consider the suspension homomorphism*

$$E: \pi_{m-1}(S^{n-1}) \rightarrow \pi_m(S^n)$$

and

$$E^\infty: \pi_{m-1}(S^{n-1}) \rightarrow \pi_{m-n}^S$$

and assume that n is even. If $m \leq n + 3$ or if $m = n + 4 \neq 8, 10$ then E and E^∞ are injective. If $m = 8$ and $n = 4$, then E is injective, but E^∞ is not.

Proof. We will use Toda's tables [T]. Since $E^\infty: \pi_5(S^3) \cong \mathbb{Z}_2 \rightarrow \pi_2^S \cong \mathbb{Z}_2$ is onto and hence bijective, it remains only to study the (nonstable) cases where $n = 4$ and $m = 7$ or 8 . The groups in the exact EHP-sequence

$$\pi_7(S^3) \xrightarrow{E} \pi_8(S^4) \xrightarrow{H} \pi_8(S^7) \xrightarrow{P} \pi_6(S^3) \xrightarrow{E} \pi_7(S^4) \longrightarrow \dots$$

(cf. [W], XII, 2.3) have order 2, 4, 2 and 12, resp.; hence both homomorphisms E are injective here. Moreover the kernel of

$$E^\infty: \pi_7(S^4) \cong \mathbb{Z} \oplus \mathbb{Z}_{12} \rightarrow \pi_3^S \cong \mathbb{Z}_{24}$$

is generated by $[\iota_4, \iota_4]$ and has a trivial intersection with $E(\pi_6(S^3)) = \{0\} \oplus \mathbb{Z}_{12}$. Our claim follows for $(m, n) = (7, 4)$ and similarly for $(m, n) = (8, 4)$ (since $\pi_7(S^3) \neq 0 = \pi_4^S$). \square

Corollary 4.6. *There exists a map $f: S^8 \rightarrow \mathbb{RP}(4)$ such that $\omega^\#(f, f) \neq 0$ but $\tilde{\omega}(f, f) = 0$. Of course the corresponding liftings $\tilde{f}: S^8 \rightarrow S^4$ have the same property.*

Proof. Indeed the groups in the exact sequence

$$\pi_8(ST(S^4)) \rightarrow \pi_8(S^4) \xrightarrow{\partial} \pi_7(S^3)$$

(cf. (1.14)) have order 2 (cf. [P]), 4 and 2, resp.; thus there is an element $[\tilde{f}] \in \pi_8(S^4)$ such that $\partial([\tilde{f}])$ and hence $\underline{\omega}^\#(\tilde{f}, \tilde{f}) = E(\partial[\tilde{f}])$ is nontrivial, but $\omega(\tilde{f}, \tilde{f}) = \tilde{\omega}(\tilde{f}, \tilde{f}) \in \pi_4^S = 0$ (compare also (4.1)). \square

Next we want to explore the fact that it is often easier to compute the stabilized versions of our ω -invariants. Indeed they just sum up the contributions of the partial coincidence manifolds C_A (not registering their linkings); also we just multiply a given bordism class with -1 if we compose the framing with a reflection of \mathbb{R}^n .

Proposition 4.7. *Given $[f] \in \pi_m(N)$, the stabilized invariant $\tilde{\omega}(f, f)$ is determined by $\underline{\tilde{\omega}}(f, f)$, i.e. by*

$$\omega(f, f) = \chi(N) \omega(f, y_0) = \omega(f, y_0) + E^\infty(\text{coll}_*([f])) = \pm E^\infty \circ \partial([f]).$$

(Here $\chi(N)$ denotes the Euler characteristic of N ; for coll_* see 3.10.) If the stable suspension homomorphism $E^\infty: \pi_{m-1}(S^{n-1}) \rightarrow \pi_{m-n}^S$ is injective (or if $m \leq n + 3$) then the conditions (i)–(v) in theorem 1.15 are all equivalent to

$$(vi) \quad \omega(f, f) = 0.$$

Proof. The first identity was already established in [K2], 2.2; the corresponding claim for $\omega^\#(f, f)$ (see [K6], 5.1) is more complicated and not so easy to use in calculations.

The second identity follows from (1.13) and 3.10. Furthermore note that $\omega(f, f) = \underline{\tilde{\omega}}(f, f)$ is the stable suspension of $\underline{\omega}^\#(f, f) = \pm E(\partial([f]))$ (cf. [K7], 5.7).

If $m \leq n + 3$ and n is even, then E^∞ is injective (cf. 4.5). If n is odd, all conditions (i)–(vi) hold automatically. \square

Proposition 4.8. *Assume that $k := \#\pi_1(N) \geq 2$. Consider arbitrary maps $f, f_1, f_2: (S^m, x_0) \rightarrow (N, y_0)$.*

If N is orientable, then

$$\chi(N) \cdot \omega(f_1, f_2) = 0 \quad (\text{cf. (3.14)})$$

and in particular

$$\chi(N) \cdot E^\infty(\text{coll}_*([f])) = 0 \quad (\text{cf. 3.10}).$$

If N is not orientable then $E^\infty(\text{coll}_([f])) = 0$ and*

$$(\chi(N) - 1) \omega(f_1, f_2) = 0;$$

if in addition $k > 2$, then $\omega(f_1, f_2) = 0$.

Proof. Note that $\omega(y_0, f) = E^\infty(\text{coll}_*([f]))$ (cf. 3.10). If N is orientable, then

$$(4.9) \quad 0 = \omega(f, f) = \omega(f, y_0) + \omega(y_0, f) = \chi(N) \cdot \omega(f, y_0)$$

(cf. 1.21, (1.13) and 4.7). Thus multiplication with the Euler characteristic annihilates also $\omega(y_0, f) = -\omega(f, y_0)$ and hence $\omega(f_1, f_2)$ (by (1.13)).

If N is not orientable, then it follows from proposition 3.12 that

$$(4.10) \quad \tilde{\omega}_A(f, y_0) = \tilde{\omega}_0(f, y_0) \quad \text{and} \quad \tilde{\omega}_A(y_0, f) = \varepsilon_A \cdot \tilde{\omega}_0(y_0, f)$$

for every $A \in \pi_1(N)$ where $\varepsilon_A = +1$ or -1 according as A is orientation preserving or reversing. Thus

$$\omega(y_0, f) = \sum_A \tilde{\omega}_A(y_0, f) = 0.$$

Moreover

$$\tilde{\omega}_A(f, f) = \tilde{\omega}_A(f, y_0) + \tilde{\omega}_A(y_0, f) = 0$$

whenever $A \neq 0$ (cf. (1.13) and (4.1)). If in addition $k > 2$ then there exist both orientation preserving and reversing $A \neq 0$ so that $\tilde{\omega}_0(f, y_0) = -\varepsilon_A \cdot \tilde{\omega}_0(y_0, f)$ both for $\varepsilon_A = +1$ and $\varepsilon_A = -1$; hence $\omega(f, y_0)$, being an even multiple of an element of order 2, vanishes, and so does $\omega(f_1, f_2)$, again by (1.13). If $\pi_1(N)$ consists only of 0 and $A \neq 0$, then

$$(4.11) \quad \tilde{\omega}_0(f, y_0) = \tilde{\omega}_A(f, y_0) = -\tilde{\omega}_A(y_0, f) = \tilde{\omega}_0(y_0, f)$$

and

$$(4.12) \quad \omega(f, y_0) = 2\tilde{\omega}_0(f, y_0) = \omega(f, f) = \chi(N) \omega(f, y_0)$$

(cf. 4.7), so that multiplication with $\chi(N) - 1$ annihilates $\omega(f, y_0)$ and hence also $\omega(f_1, f_2) = \omega(f_1, y_0)$ (cf. (4.10)). \square

Given a map $f: (S^m, x_0) \rightarrow (N, y_0)$, consider a lifting $\tilde{f}: (S^m, x_0) \rightarrow (\tilde{N}, \tilde{y}_0)$ to the universal covering space $p: \tilde{N} \rightarrow N$.

Lemma 4.13. *We have*

$$\tilde{\omega}_0(f, y_0) = \omega(\tilde{f}, \tilde{y}_0) \quad \text{and} \quad \tilde{\omega}_0(y_0, f) = \omega(\tilde{y}_0, \tilde{f}).$$

Proof. Indeed, in the Nielsen decomposition

$$f^{-1}(\{y_0\}) = \tilde{f}^{-1}(p^{-1}(\{y_0\})) = \tilde{f}^{-1}(\{A\tilde{y}_0 \mid A \in \pi_1(N)\})$$

of the relevant coincidence manifold, $\tilde{f}^{-1}(\{\tilde{y}_0\})$ is the component indexed by $A = 0$. \square

Let us apply this to the special case $\tilde{N} = S^n$ and $N = \mathbb{R}\mathbb{P}(n)$, n even. Then

$$\omega(f, f) = 2\tilde{\omega}_0(y_0, f) = 2\omega(\tilde{y}_0, \tilde{f}) = 2E^\infty([\tilde{f}])$$

(cf. (4.12), 4.13, 3.10 and (3.15)). This establishes the claim which follows theorem 1.31. Also if $m = 2n - 1$ and e.g. $n = 4, 8, 12, 14, 16$ or 20 , then $\ker E^\infty \cong \mathbb{Z}$ and E^∞ maps $\pi_m(S^n)$ onto the stable stem π_{n-1}^S which contains elements of order greater than 2 (cf. [T]). This proves the statement in example 1.26.

Similarly, (4.12) shows also (together with theorem 1.21) that $\omega(f, f) \in 2 \cdot \pi_{m-n}^S$ whenever N is not simply connected. This (together with 4.8) establishes (1.27).

Finally let us discuss example 1.28. According to fact 4.3 the Grassmann manifold $G_{5,2}$ is not orientable, 6-dimensional and its Euler characteristic equals 2. Our claim follows from proposition 4.8 and the Freudenthal suspension theorem.

5. Nielsen numbers

Definition 5.1. Given $f_1, f_2: S^m \rightarrow N$, the “strong” Nielsen number $N^\#(f_1, f_2)$ (and its stabilized analogue $N(f_1, f_2)$, resp.) is the number of elements $A \in \pi_1(N)$ such that $\omega_A^\#(f_1, f_2)$ (and $\tilde{\omega}_A(f_1, f_2)$, resp.) does not vanish (cf. (3.8) and (3.14)).

Since generic coincidence manifolds are compact these Nielsen numbers are always finite.

Proof of theorem 1.30. If $\omega^\#(f_1, f_2)$ (or $\tilde{\omega}(f_1, f_2)$, resp.) vanishes, then so do all the partial invariants $\omega_A^\#(f_1, f_2)$ (or $\tilde{\omega}_A(f_1, f_2)$, resp.), $A \in \pi_1(N)$, as well as the corresponding Nielsen number. The converse is also obvious in the stabilized setting. However, $\omega^\#(f_1, f_2)$ keeps track also of embeddings and of possible linking phenomena among the partial coincidence submanifolds $C_A(f_1, f_2)$ in S^m .

Assume that $N^\#(f_1, f_2) = 0$. Then all triples $(C_A(f_1, f_2), \tilde{g}_A^\#, \tilde{g}_A)$ (cf. (3.8)) admit individual nullbordisms in $S^m \times I$. Conceivably these can not be fitted together disjointly to yield the full *embedded* nullbordism required to show that $\omega^\#(f_1, f_2)$ vanishes. In fact, it is an open question whether the first claim in 1.30 still holds when the common domain of f_1 and f_2 is not a sphere.

But here we can use the additive structure of homotopy groups. As in (1.8) we have for all $A \in \pi_1(N)$

$$(5.2) \quad 0 = \omega_A^\#(f_1, f_2) = \omega_A^\#(f_1, f_1) + \omega_A^\#(y_0, f_2 - f_1).$$

The claim of our theorem is obvious when $\pi_1(N) = 0$ or in the selfcoincidence case since only $A = 0$ plays a rôle there (cf. (4.1)). If $A \neq 0$ then $\omega_A^\#(f_1, f_1) = 0$

and hence $\omega_A^\#(y_0, f_2 - f_1) = 0$ (cf. (5.2)). This implies – in this special root case – that the full invariant $\omega^\#(y_0, f_2 - f_1)$ vanishes; indeed, a nullbordism of $C_A(y_0, f_2 - f_1)$ gives rise to *disjoint* “parallel” nullbordisms of all the other partial coincidence manifolds $C_{A'}(y_0, f_2 - f_1)$, $A' \in \pi_1(N)$ (compare the proof of 3.12 or also [K6], 4.3). Thus in turn $\omega^\#(f_1, f_2) = \omega^\#(f_1, f_1)$ (cf. (1.8)) and we are back in the selfcoincidence case. \square

Proof of theorem 1.31. If $k = \#\pi_1(N)$ is infinite, then all pairs (f_1, f_2) are loose and hence all Nielsen numbers vanish (cf. (3.1)). If $k > 2$ or if $k = 2$ and N is orientable, then always $\omega^\#(f_1, f_1) = 0$ (cf. 1.21); therefore $\omega^\#(f_1, f_2) = \omega^\#(y_0, f_2 - f_1)$ (cf. (5.2)) and $\tilde{\omega}(f_1, f_2) = \tilde{\omega}(y_0, f_2 - f_1)$. Again 3.12 implies that the corresponding Nielsen numbers can assume only the values 0 and k in this root case. For the remaining cases compare corollary 1.25. \square

Remark 5.3. Consider the split exact sequence

$$0 \rightarrow \ker(\text{forg}) \hookrightarrow \pi_m(S^n \wedge (\Omega N)^+) \xrightarrow{\text{forg}} \bigoplus_{A \in \pi_1(N)} \pi_m(S^n \wedge A^+) \rightarrow 0 ;$$

here, given any element $\omega^\# = [C, \bar{g}^\#, \bar{g}] \in \pi_m(S^n \wedge (\Omega N)^+)$, $\text{forg}(\omega^\#)$ keeps track only of the individual A -components $[C_A = \bar{g}^{-1}(A), \bar{g}^\#|_A, \bar{g}]$ and forgets about possible linkings. As we will see below the kernel of forg can be highly nontrivial if $\#\pi_1(N) > 1$.

Consider also the function $N^\#$ which counts the nontrivial A -components. Clearly it vanishes on $\ker(\text{forg})$ and, in addition, can often assume all integer values between 0 and $\#\pi_1(N)$. This shows that theorems 1.30 and 1.31 impose strong restrictions on those $\omega^\#$ -values which can actually be realized by pairs (f_1, f_2) of maps.

Example 5.4 (real projective spaces). Consider $N = \mathbb{R}P(n)$ and its double cover $\tilde{N} = S^n$. There is a wellknown isomorphism

$$\pi_m(S^n \wedge (\Omega N)^+) \cong \pi_m(S_1^n \vee S_2^n \vee \tilde{N}, \tilde{N})$$

(cf. e.g. [K6], (61)), and the forgetful map forg (cf. 5.3) corresponds to the homomorphism

$$\pi_m(\bigvee^2 S^n \vee \tilde{N}) \rightarrow \bigoplus_{i=1}^2 \pi_m(S_i^n \vee \tilde{N})$$

which is induced by the obvious projections. Let ι_1, ι_2 and ι_0 be represented by the inclusions of the three spheres into their wedge. Then the summands of

$$\ker(\text{forg}) \cong \pi_m(S^{2n-1}) \oplus (\pi_m(S^{3n-2}))^4 \oplus \dots$$

in the Hilton decomposition of $\pi_m(S_1^n \vee S_2^n \vee \tilde{N})$ (cf. [W], XI, 6) corresponds precisely to those basic products which involve both ι_1 and ι_2 .

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