21.7 Unbestimmte Integrale

(Hinweise zur Nutzung der Tabellen s. in 8.1.1 2,**2.**, S. 445)

21.7.1 Integrale rationaler Funktionen

21.7.1.1 Integrale mit $X = ax + b$

Bezeichnung. $X = ax + b$

1 $\int X^n\,dx = \frac{1}{a(n+1)}X^{n+1} \qquad (n \neq -1),$ (für $n = -1$ s Nr 2).

2 $\int \frac{dx}{X} = \frac{1}{a}\ln X.$

3. $\int xX^n\,dx = \frac{1}{a^2(n+2)}X^{n+2} - \frac{b}{a^2(n+1)}X^{n+1}$

$(n \neq -1, \neq -2),$ (für $n = -1\,, = -2$ s. Nr 5 und 6).

4 $\int x^m X^n\,dx = \frac{1}{a^{m+1}}\int (X-b)^m X^n\,dX \qquad (n \neq -1, \neq -2, \;..,\neq -m)$

Das Integral wird für $m < n$ oder bei ganzzahligem m und gebrochenem n angewandt, in diesen Fällen wird $(X-b)^m$ nach dem binomischen Lehrsatz (s. 1.1.6.4, S. 12) entwickelt.

5 $\int \frac{x\,dx}{X} = \frac{x}{a} - \frac{b}{a^2}\ln X$

6. $\int \frac{x\,dx}{X^2} = \frac{b}{a^2X} + \frac{1}{a^2}\ln X.$

7 $\int \frac{x\,dx}{X^3} = \frac{1}{a^2}\left(-\frac{1}{X} + \frac{b}{2X^2}\right).$

8 $\int \frac{x\,dx}{X^n} = \frac{1}{a^2}\left(\frac{-1}{(n-2)X^{n-2}} + \frac{b}{(n-1)X^{n-1}}\right) \qquad (n \neq 1, \neq 2).$

9. $\int \frac{x^2\,dx}{X} = \frac{1}{a^3}\left(\frac{1}{2}X^2 - 2bX + b^2\ln X\right)$

10 $\int \frac{x^2\,dx}{X^2} = \frac{1}{a^3}\left(X - 2b\ln X - \frac{b^2}{X}\right)$

11 $\int \frac{x^2\,dx}{X^3} = \frac{1}{a^3}\left(\ln X + \frac{2b}{X} - \frac{b^2}{2X^2}\right).$

12 $\int \frac{x^2\,dx}{X^n} = \frac{1}{a^3}\left[\frac{-1}{(n-3)X^{n-3}} + \frac{2b}{(n-2)X^{n-2}} - \frac{b^2}{(n-1)X^{n-1}}\right] \qquad (n \neq 1, \neq 2, \neq 3).$

13. $\int \frac{x^3\,dx}{X} = \frac{1}{a^4}\left(\frac{X^3}{3} - \frac{3bX^2}{2} + 3b^2X - b^3\ln X\right).$

14. $\int \frac{x^3\,dx}{X^2} = \frac{1}{a^4}\left(\frac{X^2}{2} - 3bX + 3b^2\ln X + \frac{b^3}{X}\right).$

15 $\int \frac{x^3\,dx}{X^3} = \frac{1}{a^4}\left(X - 3b\ln X - \frac{3b^2}{X} + \frac{b^3}{2X^2}\right)$

16 $\displaystyle\int \frac{x^3\,dx}{X^4} = \frac{1}{a^4}\left(\ln X + \frac{3b}{X} - \frac{3b^2}{2X^2} + \frac{b^3}{3X^3}\right).$

17 $\displaystyle\int \frac{x^3\,dx}{X^n} = \frac{1}{a^4}\left[\frac{-1}{(n-4)X^{n-4}} + \frac{3b}{(n-3)X^{n-3}} - \frac{3b^2}{(n-2)X^{n-2}} + \frac{b^3}{(n-1)X^{n-1}}\right]$

$(n \neq 1, \neq 2, \neq 3, \neq 4)$

18 $\displaystyle\int \frac{dx}{xX} = -\frac{1}{b}\ln\frac{X}{x}$

19 $\displaystyle\int \frac{dx}{xX^2} = -\frac{1}{b^2}\left(\ln\frac{X}{x} + \frac{ax}{X}\right).$

20. $\displaystyle\int \frac{dx}{xX^3} = -\frac{1}{b^3}\left(\ln\frac{X}{x} + \frac{2ax}{X} - \frac{a^2x^2}{2X^2}\right).$

21 $\displaystyle\int \frac{dx}{xX^n} = -\frac{1}{b^n}\left[\ln\frac{X}{x} - \sum_{i=1}^{n-1}\binom{n-1}{i}\frac{(-a)^i x^i}{iX^i}\right] \qquad (n \geq 1).$

22. $\displaystyle\int \frac{dx}{x^2X} = -\frac{1}{bx} + \frac{a}{b^2}\ln\frac{X}{x}.$

23 $\displaystyle\int \frac{dx}{x^2X^2} = -a\left[\frac{1}{b^2X} + \frac{1}{ab^2x} - \frac{2}{b^3}\ln\frac{X}{x}\right]$

24 $\displaystyle\int \frac{dx}{x^2X^3} = -a\left[\frac{1}{2b^2X^2} + \frac{2}{b^3X} + \frac{1}{ab^3x} - \frac{3}{b^4}\ln\frac{X}{x}\right]$

25. $\displaystyle\int \frac{dx}{x^2X^n} = -\frac{1}{b^{n+1}}\left[-\sum_{i=2}^{n}\binom{n}{i}\frac{(-a)^i x^{i-1}}{(i-1)X^{i-1}} + \frac{X}{x} - na\ln\frac{X}{x}\right] \qquad (n \geq 2).$

26 $\displaystyle\int \frac{dx}{x^3X} = -\frac{1}{b^3}\left[a^2\ln\frac{X}{x} - \frac{2aX}{x} + \frac{X^2}{2x^2}\right]$

27. $\displaystyle\int \frac{dx}{x^3X^2} = -\frac{1}{b^4}\left[3a^2\ln\frac{X}{x} + \frac{a^3x}{X} + \frac{X^2}{2x^2} - \frac{3aX}{x}\right].$

28. $\displaystyle\int \frac{dx}{x^3X^3} = -\frac{1}{b^5}\left[6a^2\ln\frac{X}{x} + \frac{4a^3x}{X} - \frac{a^4x^2}{2X^2} + \frac{X^2}{2x^2} - \frac{4aX}{x}\right].$

29 $\displaystyle\int \frac{dx}{x^3X^n} = -\frac{1}{b^{n+2}}\left[-\sum_{i=3}^{n+1}\binom{n+1}{i}\frac{(-a)^i x^{i-2}}{(i-2)X^{i-2}} + \frac{a^2X^2}{2x^2} - \frac{(n+1)aX}{x}\right.$

$\displaystyle\left. + \frac{n(n+1)a^2}{2}\ln\frac{X}{x}\right] \qquad (n \geq 3)$

30. $\displaystyle\int \frac{dx}{x^mX^n} = -\frac{1}{b^{m+n-1}}\sum_{i=0}^{m+n-2}\binom{m+n-2}{i}\frac{X^{m-i-1}(-a)^i}{(m-i-1)x^{m-i-1}}.$

Wenn der Nenner des Gliedes unter dem Summenzeichen verschwindet, dann ist ein solches Glied durch das folgende zu ersetzen.

$$\binom{m+n-2}{m-1}(-a)^{m-1}\ln\frac{X}{x}$$

Bezeichnung: $\Delta = bf - ag$

31. $\int \frac{ax+b}{fx+g}\,dx = \frac{ax}{f} + \frac{\Delta}{f^2}\ln(fx+g)$

32. $\int \frac{dx}{(ax+b)(fx+g)} = \frac{1}{\Delta}\ln\frac{fx+g}{ax+b} \qquad (\Delta \neq 0).$

33. $\int \frac{x\,dx}{(ax+b)(fx+g)} = \frac{1}{\Delta}\left[\frac{b}{a}\ln(ax+b) - \frac{g}{f}\ln(fx+g)\right] \qquad (\Delta \neq 0)$

34. $\int \frac{dx}{(ax+b)^2(fx+g)} = \frac{1}{\Delta}\left(\frac{1}{ax+b} + \frac{f}{\Delta}\ln\frac{fx+g}{ax+b}\right) \qquad (\Delta \neq 0).$

35. $\int \frac{x\,dx}{(a+x)(b+x)^2} = \frac{b}{(a-b)(b+x)} - \frac{a}{(a-b)^2}\ln\frac{a+x}{b+x} \qquad (a \neq b).$

36 $\int \frac{x^2\,dx}{(a+x)(b+x)^2} = \frac{b^2}{(b-a)(b+x)} + \frac{a^2}{(b-a)^2}\ln(a+x) + \frac{b^2-2ab}{(b-a)^2}\ln(b+x) \qquad (a \neq b)$

37 $\int \frac{dx}{(a+x)^2(b+x)^2} = \frac{-1}{(a-b)^2}\left(\frac{1}{a+x} + \frac{1}{b+x}\right) + \frac{2}{(a-b)^3}\ln\frac{a+x}{b+x} \qquad (a \neq b).$

38 $\int \frac{x\,dx}{(a+x)^2(b+x)^2} = \frac{1}{(a-b)^2}\left(\frac{a}{a+x} + \frac{b}{b+x}\right) + \frac{a+b}{(a-b)^3}\ln\frac{a+x}{b+x} \qquad (a \neq b).$

39 $\int \frac{x^2\,dx}{(a+x)^2(b+x)^2} = \frac{-1}{(a-b)^2}\left(\frac{a^2}{a+x} + \frac{b^2}{b+x}\right) + \frac{2ab}{(a-b)^3}\ln\frac{a+x}{b+x} \qquad (a \neq b).$

21.7.1.2 Integrale mit $X = ax^2 + bx + c$

Bezeichnungen: $X = ax^2 + bx + c$; $\Delta = 4ac - b^2$

40 $\int \frac{dx}{X} = \frac{2}{\sqrt{\Delta}}\arctan\frac{2ax+b}{\sqrt{\Delta}} \qquad$ (fur $\Delta > 0$),

$= -\frac{2}{\sqrt{-\Delta}}\operatorname{Artanh}\frac{2ax+b}{\sqrt{-\Delta}} \qquad$ (für $\Delta < 0$),

$= \frac{1}{\sqrt{-\Delta}}\ln\frac{2ax+b-\sqrt{-\Delta}}{2ax+b+\sqrt{-\Delta}} \qquad$ (für $\Delta < 0$).

41. $\int \frac{dx}{X^2} = \frac{2ax+b}{\Delta X} + \frac{2a}{\Delta}\int\frac{dx}{X}$ (s Nr 40).

42 $\int \frac{dx}{X^3} = \frac{2ax+b}{\Delta}\left(\frac{1}{2X^2} + \frac{3a}{\Delta X}\right) + \frac{6a^2}{\Delta^2}\int\frac{dx}{X}$ (s Nr.40).

43. $\int \frac{dx}{X^n} = \frac{2ax+b}{(n-1)\Delta X^{n-1}} + \frac{(2n-3)2a}{(n-1)\Delta}\int\frac{dx}{X^{n-1}}.$

44. $\displaystyle\int \frac{x\,dx}{X} = \frac{1}{2a}\ln X - \frac{b}{2a}\int\frac{dx}{X}$ (s Nr 40)

45. $\displaystyle\int \frac{x\,dx}{X^2} = -\frac{bx+2c}{\Delta X} - \frac{b}{\Delta}\int\frac{dx}{X}$ (s Nr 40)

46. $\displaystyle\int \frac{x\,dx}{X^n} = -\frac{bx+2c}{(n-1)\Delta X^{n-1}} - \frac{b(2n-3)}{(n-1)\Delta}\int\frac{dx}{X^{n-1}}.$

47. $\displaystyle\int \frac{x^2\,dx}{X} = \frac{x}{a} - \frac{b}{2a^2}\ln X + \frac{b^2-2ac}{2a^2}\int\frac{dx}{X}$ (s Nr 40)

48. $\displaystyle\int \frac{x^2\,dx}{X^2} = \frac{(b^2-2ac)x+bc}{a\Delta X} + \frac{2c}{\Delta}\int\frac{dx}{X}$ (s Nr 40)

49. $\displaystyle\int \frac{x^2\,dx}{X^n} = \frac{-x}{(2n-3)aX^{n-1}} + \frac{c}{(2n-3)a}\int\frac{dx}{X^n} - \frac{(n-2)b}{(2n-3)a}\int\frac{x\,dx}{X^n}$ (s. Nr 43 u. 46)

50. $\displaystyle\int \frac{x^m\,dx}{X^n} = -\frac{x^{m-1}}{(2n-m-1)aX^{n-1}} + \frac{(m-1)c}{(2n-m-1)a}\int\frac{x^{m-2}\,dx}{X^n}$

$\displaystyle -\frac{(n-m)b}{(2n-m-1)a}\int\frac{x^{m-1}\,dx}{X^n} \quad (m \neq 2n-1);$ (für $m = 2n-1$ s. Nr.51)

51. $\displaystyle\int \frac{x^{2n-1}\,dx}{X^n} = \frac{1}{a}\int\frac{x^{2n-3}\,dx}{X^{n-1}} - \frac{c}{a}\int\frac{x^{2n-3}\,dx}{X^n} - \frac{b}{a}\int\frac{x^{2n-2}\,dx}{X^n}.$

52. $\displaystyle\int \frac{dx}{xX} = \frac{1}{2c}\ln\frac{x^2}{X} - \frac{b}{2c}\int\frac{dx}{X}$ (s. Nr 40)

53 $\displaystyle\int \frac{dx}{xX^n} = \frac{1}{2c(n-1)X^{n-1}} - \frac{b}{2c}\int\frac{dx}{X^n} + \frac{1}{c}\int\frac{dx}{xX^{n-1}}.$

54 $\displaystyle\int \frac{dx}{x^2X} = \frac{b}{2c^2}\ln\frac{X}{x^2} - \frac{1}{cx} + \left(\frac{b^2}{2c^2} - \frac{a}{c}\right)\int\frac{dx}{X}$ (s. Nr 40).

55. $\displaystyle\int \frac{dx}{x^mX^n} = -\frac{1}{(m-1)cx^{m-1}X^{n-1}} - \frac{(2n+m-3)a}{(m-1)c}\int\frac{dx}{x^{m-2}X^n}$

$\displaystyle -\frac{(n+m-2)b}{(m-1)c}\int\frac{dx}{x^{m-1}X^n}$ $(m > 1)$

56. $\displaystyle\int \frac{dx}{(fx+g)X} = \frac{1}{2(cf^2-gbf+g^2a)}\left[f\ln\frac{(fx+g)^2}{X}\right]$

$\displaystyle +\frac{2ga-bf}{2(cf^2-gbf+g^2a)}\int\frac{dx}{X}$ (s. Nr 40)

21.7.1.3 Integrale mit $X = a^2 \pm x^2$

Bezeichnungen: $X = a^2 \pm x^2$,

$$Y = \begin{cases} \arctan\dfrac{x}{a} & \text{für das Vorzeichen „ + “,} \\ \operatorname{Artanh}\dfrac{x}{a} = \dfrac{1}{2}\ln\dfrac{a+x}{a-x} & \text{für das Vorzeichen „ − “ und } |x| < a, \\ \operatorname{Arcoth}\dfrac{x}{a} = \dfrac{1}{2}\ln\dfrac{x+a}{x-a} & \text{für das Vorzeichen „ − “ und } |x| > a \end{cases}$$

Im Falle eines Doppelvorzeichens in einer Formel gehört das obere Vorzeichen zu $X = a^2 + x^2$, das untere zu $X = a^2 - x^2$, $a > 0$

57. $\int \frac{dx}{X} = \frac{1}{a} Y$

58. $\int \frac{dx}{X^2} = \frac{x}{2a^2 X} + \frac{1}{2a^3} Y.$

59. $\int \frac{dx}{X^3} = \frac{x}{4a^2 X^2} + \frac{3x}{8a^4 X} + \frac{3}{8a^5} Y.$

60. $\int \frac{dx}{X^{n+1}} = \frac{x}{2na^2 X^n} + \frac{2n-1}{2na^2} \int \frac{dx}{X^n}.$

61. $\int \frac{x\,dx}{X} = \pm \frac{1}{2} \ln X.$

62. $\int \frac{x\,dx}{X^2} = \mp \frac{1}{2X}$

63. $\int \frac{x\,dx}{X^3} = \mp \frac{1}{4X^2}$

64. $\int \frac{x\,dx}{X^{n+1}} = \mp \frac{1}{2nX^n} \qquad (n \neq 0).$

65. $\int \frac{x^2\,dx}{X} = \pm x \mp aY.$

66. $\int \frac{x^2\,dx}{X^2} = \mp \frac{x}{2X} \pm \frac{1}{2a} Y.$

67. $\int \frac{x^2\,dx}{X^3} = \mp \frac{x}{4X^2} \pm \frac{x}{8a^2 X} \pm \frac{1}{8a^3} Y.$

68. $\int \frac{x^2\,dx}{X^{n+1}} = \mp \frac{x}{2nX^n} \pm \frac{1}{2n} \int \frac{dx}{X^n} \qquad (n \neq 0).$

69. $\int \frac{x^3\,dx}{X} = \pm \frac{x^2}{2} - \frac{a^2}{2} \ln X$

70. $\int \frac{x^3\,dx}{X^2} = \frac{a^2}{2X} + \frac{1}{2} \ln X.$

71. $\int \frac{x^3\,dx}{X^3} = -\frac{1}{2X} + \frac{a^2}{4X^2}.$

72. $\int \frac{x^3\,dx}{X^{n+1}} = -\frac{1}{2(n-1)X^{n-1}} + \frac{a^2}{2nX^n} \qquad (n > 1).$

73. $\int \frac{dx}{xX} = \frac{1}{2a^2} \ln \frac{x^2}{X}.$

74. $\int \frac{dx}{xX^2} = \frac{1}{2a^2 X} + \frac{1}{2a^4} \ln \frac{x^2}{X}.$

75. $\int \frac{dx}{xX^3} = \frac{1}{4a^2 X^2} + \frac{1}{2a^4 X} + \frac{1}{2a^6} \ln \frac{x^2}{X}.$

76. $\int \frac{dx}{x^2 X} = -\frac{1}{a^2 x} \mp \frac{1}{a^3} Y.$

77. $\int \frac{dx}{x^2X^2} = -\frac{1}{a^4x} \mp \frac{x}{2a^4X} \mp \frac{3}{2a^5}Y$

78 $\int \frac{dx}{x^2X^3} = -\frac{1}{a^6x} \mp \frac{x}{4a^4X^2} \mp \frac{7x}{8a^6X} \mp \frac{15}{8a^7}Y$

79. $\int \frac{dx}{x^3X} = -\frac{1}{2a^2x^2} \mp \frac{1}{2a^4} \ln \frac{x^2}{X}$

80 $\int \frac{dx}{x^3X^2} = -\frac{1}{2a^4x^2} \mp \frac{1}{2a^4X} \mp \frac{1}{a^6} \ln \frac{x^2}{X}$

81 $\int \frac{dx}{x^3X^3} = -\frac{1}{2a^6x^2} \mp \frac{1}{a^6X} \mp \frac{1}{4a^4X^2} \mp \frac{3}{2a^8} \ln \frac{x^2}{X}$

82 $\int \frac{dx}{(b+cx)X} = \frac{1}{a^2c^2 \pm b^2}\left[c\ln(b+cx) - \frac{c}{2}\ln X \pm \frac{b}{a}Y\right]$

21.7.1.4 Integrale mit $X = a^3 \pm x^3$

Bezeichnungen. $a^3 \pm x^3 = X$: im Falle eines Doppelvorzeichens in einer Formel gehört das obere Vorzeichen zu $X = a^3 + x^3$, das untere zu $X = a^3 - x^3$

83 $\int \frac{dx}{X} = \pm\frac{1}{6a^2} \ln \frac{(a \pm x)^2}{a^2 \mp ax + x^2} + \frac{1}{a^2\sqrt{3}} \arctan \frac{2x \mp a}{a\sqrt{3}}$

84 $\int \frac{dx}{X^2} = \frac{x}{3a^3X} + \frac{2}{3a^3}\int \frac{dx}{X}$ (s Nr 83)

85 $\int \frac{x\,dx}{X} = \frac{1}{6a} \ln \frac{a^2 \mp ax + x^2}{(a \pm x^2)} \pm \frac{1}{a\sqrt{3}} \arctan \frac{2x \mp a}{a\sqrt{3}}.$

86 $\int \frac{x\,dx}{X^2} = \frac{x^2}{3a^3X} + \frac{1}{3a^3}\int \frac{x\,dx}{X}$ (s. Nr 85)

87 $\int \frac{x^2\,dx}{X} = \pm\frac{1}{3}\ln X$

88 $\int \frac{x^2\,dx}{X^2} = \mp\frac{1}{3X}.$

89. $\int \frac{x^3\,dx}{X} = \pm x \mp a^3 \int \frac{dx}{X}$ (s. Nr.83)

90 $\int \frac{x^3\,dx}{X^2} = \mp\frac{x}{3X} \pm \frac{1}{3}\int \frac{dx}{X}$ (s Nr 83)

91. $\int \frac{dx}{xX} = \frac{1}{3a^3} \ln \frac{x^3}{X}.$

92 $\int \frac{dx}{xX^2} = \frac{1}{3a^3X} + \frac{1}{3a^6} \ln \frac{x^3}{X}.$

93. $\int \frac{dx}{x^2X} = -\frac{1}{a^3x} \mp \frac{1}{a^3}\int \frac{x\,dx}{X}$ (s Nr 85)

94. $\int \frac{dx}{x^2X^2} = -\frac{1}{a^6x} \mp \frac{x^2}{3a^6X} \mp \frac{4}{3a^6}\int \frac{x\,dx}{X}$ (s Nr 85)

95 $\int \frac{dx}{x^3 X} = -\frac{1}{2a^3 x^2} \mp \frac{1}{a^3} \int \frac{dx}{X}$ (s. Nr 83)

96 $\int \frac{dx}{x^3 X^2} = -\frac{1}{2a^6 x^2} \mp \frac{x}{3a^6 X} \mp \frac{5}{3a^6} \int \frac{dx}{X}$ (s. Nr 83)

21.7.1.5 Integrale mit $X = a^4 + x^4$

97 $\int \frac{dx}{a^4 + x^4} = \frac{1}{4a^3\sqrt{2}} \ln \frac{x^2 + ax\sqrt{2} + a^2}{x^2 - ax\sqrt{2} + a^2} + \frac{1}{2a^3\sqrt{2}} \arctan \frac{ax\sqrt{2}}{a^2 - x^2}$

98 $\int \frac{x\,dx}{a^4 + x^4} = \frac{1}{2a^2} \arctan \frac{x^2}{a^2}.$

99 $\int \frac{x^2\,dx}{a^4 + x^4} = -\frac{1}{4a\sqrt{2}} \ln \frac{x^2 + ax\sqrt{2} + a^2}{x^2 - ax\sqrt{2} + a^2} + \frac{1}{2a\sqrt{2}} \arctan \frac{ax\sqrt{2}}{a^2 - x^2}$

100 $\int \frac{x^3\,dx}{a^4 + x^4} = \frac{1}{4} \ln(a^4 + x^4)$

21.7.1.6 Integrale mit $X = a^4 - x^4$

101 $\int \frac{dx}{a^4 - x^4} = \frac{1}{4a^3} \ln \frac{a + x}{a - x} + \frac{1}{2a^3} \arctan \frac{x}{a}$

102 $\int \frac{x\,dx}{a^4 - x^4} = \frac{1}{4a^3} \ln \frac{a^2 + x^2}{a^2 - x^2}.$

103 $\int \frac{x^2\,dx}{a^4 - x^4} = \frac{1}{4a} \ln \frac{a + x}{a - x} - \frac{1}{2a} \arctan \frac{x}{a}.$

104. $\int \frac{x^3\,dx}{a^4 - x^4} = -\frac{1}{4} \ln(a^4 - x^4).$

21.7.1.7 Einige Fälle der Partialbruchzerlegung

105 $\frac{1}{(a + bx)(f + gx)} \equiv \frac{1}{fb - ag} \left(\frac{b}{a + bx} - \frac{g}{f + gx} \right)$

106 $\frac{1}{(x + a)(x + b)(x + c)} \equiv \frac{A}{x + a} + \frac{B}{x + b} + \frac{C}{x + c},$

wobei gilt $A = \frac{1}{(b - a)(c - a)},\ B = \frac{1}{(a - b)(c - b)},\ C = \frac{1}{(a - c)(b - c)}$

107 $\frac{1}{(x + a)(x + b)(x + c)(x + d)} \equiv \frac{A}{x + a} + \frac{B}{x + b} + \frac{C}{x + c} + \frac{D}{x + d},$

wobei gilt $A = \frac{1}{(b - a)(c - a)(d - a)},\ B = \frac{1}{(a - b)(c - b)(d - b)}$ usw

108. $\frac{1}{(a + bx^2)(f + gx^2)} \equiv \frac{1}{fb - ag} \left(\frac{b}{a + bx^2} - \frac{g}{f + gx^2} \right)$

21.7.2 Integrale irrationaler Funktionen

21.7.2.1 Integrale mit $\sqrt{x}$ und $a^2 \pm b^2x$

Bezeichnungen:

$$X = a^2 \pm b^2 x\,,\ Y = \begin{cases} \arctan \dfrac{b\sqrt{x}}{a} & \text{für das Vorzeichen „ + “,} \\ \dfrac{1}{2}\ln \dfrac{a + b\sqrt{x}}{a - b\sqrt{x}} & \text{für das Vorzeichen „ − “.} \end{cases}$$

Im Falle eines Doppelvorzeichens in einer Formel gehört das obere Vorzeichen zu $X = a^2 + b^2x$, das untere zu $X = a^2 - b^2x$.

109. $\displaystyle\int \frac{\sqrt{x}\,dx}{X} = \pm\frac{2\sqrt{x}}{b^2} \mp \frac{2a}{b^3}Y$

110. $\displaystyle\int \frac{\sqrt{x^3}\,dx}{X} = \pm\frac{2}{3}\frac{\sqrt{x^3}}{b^2} - \frac{2a^2\sqrt{x}}{b^4} + \frac{2a^3}{b^5}Y.$

111 $\displaystyle\int \frac{\sqrt{x}\,dx}{X^2} = \mp\frac{\sqrt{x}}{b^2X} \pm \frac{1}{ab^3}Y.$

112. $\displaystyle\int \frac{\sqrt{x^3}\,dx}{X^2} = \pm\frac{2\sqrt{x^3}}{b^2X} + \frac{3a^2\sqrt{x}}{b^4X} - \frac{3a}{b^5}Y$

113 $\displaystyle\int \frac{dx}{X\sqrt{x}} = \frac{2}{ab}Y.$

114. $\displaystyle\int \frac{dx}{X\sqrt{x^3}} = -\frac{2}{a^2\sqrt{x}} \mp \frac{2b}{a^3}Y$

115 $\displaystyle\int \frac{dx}{X^2\sqrt{x}} = \frac{\sqrt{x}}{a^2X} + \frac{1}{a^3b}Y.$

116. $\displaystyle\int \frac{dx}{X^2\sqrt{x^3}} = -\frac{2}{a^2X\sqrt{x}} \mp \frac{3b^2\sqrt{x}}{a^4X} \mp \frac{3b}{a^5}Y.$

21.7.2.2 Andere Integrale mit $\sqrt{x}$

117. $\displaystyle\int \frac{\sqrt{x}\,dx}{a^4 + x^2} = -\frac{1}{2a\sqrt{2}}\ln\frac{x + a\sqrt{2x} + a^2}{x - a\sqrt{2x} + a^2} + \frac{1}{a\sqrt{2}}\arctan\frac{a\sqrt{2x}}{a^2 - x}.$

118. $\displaystyle\int \frac{dx}{(a^4 + x^2)\sqrt{x}} = \frac{1}{2a^3\sqrt{2}}\ln\frac{x + a\sqrt{2x} + a^2}{x - a\sqrt{2x} + a^2} + \frac{1}{a^3\sqrt{2}}\arctan\frac{a\sqrt{2x}}{a^2 - x}.$

119. $\displaystyle\int \frac{\sqrt{x}\,dx}{a^4 - x^2} = \frac{1}{2a}\ln\frac{a + \sqrt{x}}{a - \sqrt{x}} - \frac{1}{a}\arctan\frac{\sqrt{x}}{a}.$

120 $\displaystyle\int \frac{dx}{(a^4 - x^2)\sqrt{x}} = \frac{1}{2a^3}\ln\frac{a + \sqrt{x}}{a - \sqrt{x}} + \frac{1}{a^3}\arctan\frac{\sqrt{x}}{a}.$

21.7.2.3 Integrale mit $\sqrt{ax + b}$

Bezeichnung: $X = ax + b$

121 $\int \sqrt{X}\,dx = \frac{2}{3a}\sqrt{X^3}$

122. $\int x\sqrt{X}\,dx = \frac{2(3ax-2b)\sqrt{X^3}}{15a^2}$

123 $\int x^2\sqrt{X}\,dx = \frac{2(15a^2x^2-12abx+8b^2)\sqrt{X^3}}{105a^3}$

124 $\int \frac{dx}{\sqrt{X}} = \frac{2\sqrt{X}}{a}$

125 $\int \frac{x\,dx}{\sqrt{X}} = \frac{2(ax-2b)}{3a^2}\sqrt{X}.$

126 $\int \frac{x^2\,dx}{\sqrt{X}} = \frac{2(3a^2x^2-4abx+8b^2)\sqrt{X}}{15a^3}$

127 $\int \frac{dx}{x\sqrt{X}} = \begin{cases} -\frac{2}{\sqrt{b}}\operatorname{Arcoth}\sqrt{\frac{X}{b}} = \frac{1}{\sqrt{b}}\ln\frac{\sqrt{X}-\sqrt{b}}{\sqrt{X}+\sqrt{b}} & \text{für } b>0\,, \\ \frac{2}{\sqrt{-b}}\arctan\sqrt{\frac{X}{-b}} & \text{für } b<0 \end{cases}$

128 $\int \frac{\sqrt{X}}{x}dx = 2\sqrt{X} + b\int\frac{dx}{x\sqrt{X}}$ (s. Nr 127)

129 $\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{bx} - \frac{a}{2b}\int\frac{dx}{x\sqrt{X}}$ (s Nr 127)

130 $\int \frac{\sqrt{X}}{x^2}dx = -\frac{\sqrt{X}}{x} + \frac{a}{2}\int\frac{dx}{x\sqrt{X}}$ (s. Nr. 127)

131 $\int \frac{dx}{x^n\sqrt{X}} = -\frac{\sqrt{X}}{(n-1)bx^{n-1}} - \frac{(2n-3)a}{(2n-2)b}\int\frac{dx}{x^{n-1}\sqrt{X}}.$

132 $\int \sqrt{X^3}\,dx = \frac{2\sqrt{X^5}}{5a}$

133 $\int x\sqrt{X^3}\,dx = \frac{2}{35a^2}\left(5\sqrt{X^7} - 7b\sqrt{X^5}\right)$

134 $\int x^2\sqrt{X^3}\,dx = \frac{2}{a^3}\left(\frac{\sqrt{X^9}}{9} - \frac{2b\sqrt{X^7}}{7} + \frac{b^2\sqrt{X^5}}{5}\right)$

135 $\int \frac{\sqrt{X^3}}{x}dx = \frac{2\sqrt{X^3}}{3} + 2b\sqrt{X} + b^2\int\frac{dx}{x\sqrt{X}}$ (s Nr. 127)

136 $\int \frac{x\,dx}{\sqrt{X^3}} = \frac{2}{a^2}\left(\sqrt{X} + \frac{b}{\sqrt{X}}\right)$

137 $\int \frac{x^2\,dx}{\sqrt{X^3}} = \frac{2}{a^3}\left(\frac{\sqrt{X^3}}{3} - 2b\sqrt{X} - \frac{b^2}{\sqrt{X}}\right)$

138. $\displaystyle\int \frac{dx}{x\sqrt{X^3}} = \frac{2}{b\sqrt{X}} + \frac{1}{b}\int \frac{dx}{x\sqrt{X}}$ (s. Nr. 127)

139 $\displaystyle\int \frac{dx}{x^2\sqrt{X^3}} = -\frac{1}{bx\sqrt{X}} - \frac{3a}{b^2\sqrt{X}} - \frac{3a}{2b^2}\int \frac{dx}{x\sqrt{X}}$ (s. Nr. 127).

140 $\displaystyle\int X^{\pm n/2}\,dx = \frac{2X^{(2\pm n)/2}}{a(2\pm n)}$

141. $\displaystyle\int xX^{\pm n/2}\,dx = \frac{2}{a^2}\left(\frac{X^{(4\pm n)/2}}{4\pm n} - \frac{bX^{(2\pm n)/2}}{2\pm n}\right).$

142 $\displaystyle\int x^2X^{\pm n/2}\,dx = \frac{2}{a^3}\left(\frac{X^{(6\pm n)/2}}{6\pm n} - \frac{2bX^{(4\pm n)/2}}{4\pm n} + \frac{b^2X^{(2\pm n)/2}}{2\pm n}\right).$

143 $\displaystyle\int \frac{X^{n/2}\,dx}{x} = \frac{2X^{n/2}}{n} + b\int \frac{X^{(n-2)/2}}{x}dx.$

144 $\displaystyle\int \frac{dx}{xX^{n/2}} = \frac{2}{(n-2)bX^{(n-2)/2}} + \frac{1}{b}\int \frac{dx}{xX^{(n-2)/2}}.$

145 $\displaystyle\int \frac{dx}{x^2X^{n/2}} = -\frac{1}{bxX^{(n-2)/2}} - \frac{na}{2b}\int \frac{dx}{xX^{n/2}}.$

21.7.2.4 Integrale mit $\sqrt{ax+b}$ und $\sqrt{fx+g}$

Bezeichnungen: $X = ax + b$, $Y = fx + g$, $\Delta = bf - ag$

146. $\displaystyle\int \frac{dx}{\sqrt{XY}} = \begin{cases} -\frac{2}{\sqrt{-af}}\arctan\sqrt{-\frac{fX}{aY}} & \text{für } af < 0, \\ \frac{2}{\sqrt{af}}\operatorname{Artanh}\sqrt{\frac{fX}{aY}} & \text{für } af > 0, \\ \frac{2}{\sqrt{af}}\ln\left(\sqrt{aY}+\sqrt{fX}\right) & \text{für } af > 0. \end{cases}$

147. $\displaystyle\int \frac{x\,dx}{\sqrt{XY}} = \frac{\sqrt{XY}}{af} - \frac{ag+bf}{2af}\int \frac{dx}{\sqrt{XY}}$ (s. Nr. 146).

148. $\displaystyle\int \frac{dx}{\sqrt{X}\sqrt{Y^3}} = -\frac{2\sqrt{X}}{\Delta\sqrt{Y}}.$

149. $\displaystyle\int \frac{dx}{Y\sqrt{X}} = \begin{cases} \frac{2}{\sqrt{-\Delta f}}\arctan\frac{f\sqrt{X}}{\sqrt{-\Delta f}} & \text{für } \Delta f < 0, \\ \frac{1}{\sqrt{\Delta f}}\ln\frac{f\sqrt{X}-\sqrt{\Delta f}}{f\sqrt{X}+\sqrt{\Delta f}} & \text{fur } \Delta f > 0. \end{cases}$

150. $\displaystyle\int \sqrt{XY}\,dx = \frac{\Delta+2aY}{4af}\sqrt{XY} - \frac{\Delta^2}{8af}\int \frac{dx}{\sqrt{XY}}$ (s. Nr. 146).

151 $\displaystyle\int \sqrt{\frac{Y}{X}}dx = \frac{1}{a}\sqrt{XY} - \frac{\Delta}{2a}\int \frac{dx}{\sqrt{XY}}$ (s. Nr 146)

152 $\int \frac{\sqrt{X}\,dx}{Y} = \frac{2\sqrt{X}}{f} + \frac{\Delta}{f}\int \frac{dx}{Y\sqrt{X}}$ (s Nr 149)

153 $\int \frac{Y^n\,dx}{\sqrt{X}} = \frac{2}{(2n+1)a}\left(\sqrt{X}Y^n - n\Delta\int \frac{Y^{n-1}\,dx}{\sqrt{X}}\right)$

154 $\int \frac{dx}{\sqrt{X}Y^n} = -\frac{1}{(n-1)\Delta}\left\{\frac{\sqrt{X}}{Y^{n-1}} + \left(n - \frac{3}{2}\right)a\int \frac{dx}{\sqrt{X}Y^{n-1}}\right\}.$

155 $\int \sqrt{X}Y^n\,dx = \frac{1}{(2n+3)f}\left(2\sqrt{X}Y^{n+1} + \Delta\int \frac{Y^n\,dx}{\sqrt{X}}\right)$ (s Nr 153)

156 $\int \frac{\sqrt{X}\,dx}{Y^n} = \frac{1}{(n-1)f}\left(-\frac{\sqrt{X}}{Y^{n-1}} + \frac{a}{2}\int \frac{dx}{\sqrt{X}Y^{n-1}}\right).$

21.7.2.5 Integrale mit $\sqrt{a^2 - x^2}$

Bezeichnung: $X = a^2 - x^2$

157 $\int \sqrt{X}\,dx = \frac{1}{2}\left(x\sqrt{X} + a^2 \arcsin\frac{x}{a}\right).$

158 $\int x\sqrt{X}\,dx = -\frac{1}{3}\sqrt{X^3}$

159 $\int x^2\sqrt{X}\,dx = -\frac{x}{4}\sqrt{X^3} + \frac{a^2}{8}\left(x\sqrt{X} + a^2\arcsin\frac{x}{a}\right)$

160 $\int x^3\sqrt{X}\,dx = \frac{\sqrt{X^5}}{5} - a^2\frac{\sqrt{X^3}}{3}$

161 $\int \frac{\sqrt{X}}{x}dx = \sqrt{X} - a\ln\frac{a+\sqrt{X}}{x}.$

162 $\int \frac{\sqrt{X}}{x^2}dx = -\frac{\sqrt{X}}{x} - \arcsin\frac{x}{a}.$

163 $\int \frac{\sqrt{X}}{x^3}dx = -\frac{\sqrt{X}}{2x^2} + \frac{1}{2a}\ln\frac{a+\sqrt{X}}{x}.$

164 $\int \frac{dx}{\sqrt{X}} = \arcsin\frac{x}{a}$

165 $\int \frac{x\,dx}{\sqrt{X}} = -\sqrt{X}$

166 $\int \frac{x^2\,dx}{\sqrt{X}} = -\frac{x}{2}\sqrt{X} + \frac{a^2}{2}\arcsin\frac{x}{a}$

167 $\int \frac{x^3\,dx}{\sqrt{X}} = \frac{\sqrt{X^3}}{3} - a^2\sqrt{X}.$

168 $\int \frac{dx}{x\sqrt{X}} = -\frac{1}{a}\ln\frac{a+\sqrt{X}}{x}$

169 $$\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{a^2x}.$$

170. $$\int \frac{dx}{x^3\sqrt{X}} = -\frac{\sqrt{X}}{2a^2x^2} - \frac{1}{2a^3}\ln\frac{a+\sqrt{X}}{x}.$$

171 $$\int \sqrt{X^3}\,dx = \frac{1}{4}\left(x\sqrt{X^3} + \frac{3a^2x}{2}\sqrt{X} + \frac{3a^4}{2}\arcsin\frac{x}{a}\right)$$

172 $$\int x\sqrt{X^3}\,dx = -\frac{1}{5}\sqrt{X^5}.$$

173 $$\int x^2\sqrt{X^3}\,dx = -\frac{x\sqrt{X^5}}{6} + \frac{a^2x\sqrt{X^3}}{24} + \frac{a^4x\sqrt{X}}{16} + \frac{a^6}{16}\arcsin\frac{x}{a}.$$

174 $$\int x^3\sqrt{X^3}\,dx = \frac{\sqrt{X^7}}{7} - \frac{a^2\sqrt{X^5}}{5}.$$

175. $$\int \frac{\sqrt{X^3}}{x}\,dx = \frac{\sqrt{X^3}}{3} + a^2\sqrt{X} - a^3\ln\frac{a+\sqrt{X}}{x}.$$

176 $$\int \frac{\sqrt{X^3}}{x^2}\,dx = -\frac{\sqrt{X^3}}{x} - \frac{3}{2}x\sqrt{X} - \frac{3}{2}a^2\arcsin\frac{x}{a}.$$

177. $$\int \frac{\sqrt{X^3}}{x^3}\,dx = -\frac{\sqrt{X^3}}{2x^2} - \frac{3\sqrt{X}}{2} + \frac{3a}{2}\ln\frac{a+\sqrt{X}}{x}$$

178. $$\int \frac{dx}{\sqrt{X^3}} = \frac{x}{a^2\sqrt{X}}.$$

179. $$\int \frac{x\,dx}{\sqrt{X^3}} = \frac{1}{\sqrt{X}}.$$

180 $$\int \frac{x^2\,dx}{\sqrt{X^3}} = \frac{x}{\sqrt{X}} - \arcsin\frac{x}{a}$$

181 $$\int \frac{x^3\,dx}{\sqrt{X^3}} = \sqrt{X} + \frac{a^2}{\sqrt{X}}.$$

182 $$\int \frac{dx}{x\sqrt{X^3}} = \frac{1}{a^2\sqrt{X}} - \frac{1}{a^3}\ln\frac{a+\sqrt{X}}{x}.$$

183 $$\int \frac{dx}{x^2\sqrt{X^3}} = \frac{1}{a^4}\left(-\frac{\sqrt{X}}{x} + \frac{x}{\sqrt{X}}\right)$$

184 $$\int \frac{dx}{x^3\sqrt{X^3}} = -\frac{1}{2a^2x^2\sqrt{X}} + \frac{3}{2a^4\sqrt{X}} - \frac{3}{2a^5}\ln\frac{a+\sqrt{X}}{x}.$$

21.7.2.6 Integrale mit $\sqrt{x^2 + a^2}$

Bezeichnung: $X = x^2 + a^2$

185. $$\int \sqrt{X}\,dx = \frac{1}{2}\left(x\sqrt{X} + a^2\operatorname{Arsinh}\frac{x}{a}\right) + C = \frac{1}{2}\left[x\sqrt{X} + a^2\ln\left(x+\sqrt{X}\right)\right] + C_1\,.$$

186 $\int x\sqrt{X}\,dx = \frac{1}{3}\sqrt{X^3}$.

187 $\int x^2\sqrt{X}\,dx = \frac{x}{4}\sqrt{X^3} - \frac{a^2}{8}\left(x\sqrt{X} + a^2 \operatorname{Arsinh}\frac{x}{a}\right) + C$
$= \frac{x}{4}\sqrt{X^3} - \frac{a^2}{8}\left[x\sqrt{X} + a^2 \ln\left(x + \sqrt{X}\right)\right] + C_1$.

188 $\int x^3\sqrt{X}\,dx = \frac{\sqrt{X^5}}{5} - \frac{a^2\sqrt{X^3}}{3}$.

189 $\int \frac{\sqrt{X}}{x}\,dx = \sqrt{X} - a\ln\frac{a+\sqrt{X}}{x}$

190. $\int \frac{\sqrt{X}}{x^2}\,dx = -\frac{\sqrt{X}}{x} + \operatorname{Arsinh}\frac{x}{a} + C = -\frac{\sqrt{X}}{x} + \ln\left(x + \sqrt{X}\right) + C_1$

191 $\int \frac{\sqrt{X}}{x^3}\,dx = -\frac{\sqrt{X}}{2x^2} - \frac{1}{2a}\ln\frac{a+\sqrt{X}}{x}$

192 $\int \frac{dx}{\sqrt{X}} = \operatorname{Arsinh}\frac{x}{a} + C = \ln\left(x + \sqrt{X}\right) + C_1$

193 $\int \frac{x\,dx}{\sqrt{X}} = \sqrt{X}$

194. $\int \frac{x^2\,dx}{\sqrt{X}} = \frac{x}{2}\sqrt{X} - \frac{a^2}{2}\operatorname{Arsinh}\frac{x}{a} + C = \frac{x}{2}\sqrt{X} - \frac{a^2}{2}\ln\left(x + \sqrt{X}\right) + C_1$

195. $\int \frac{x^3\,dx}{\sqrt{X}} = \frac{\sqrt{X^3}}{3} - a^2\sqrt{X}$

196 $\int \frac{dx}{x\sqrt{X}} = -\frac{1}{a}\ln\frac{a+\sqrt{X}}{x}$

197 $\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{a^2x}$.

198 $\int \frac{dx}{x^3\sqrt{X}} = -\frac{\sqrt{X}}{2a^2x^2} + \frac{1}{2a^3}\ln\frac{a+\sqrt{X}}{x}$.

199 $\int \sqrt{X^3}\,dx = \frac{1}{4}\left(x\sqrt{X^3} + \frac{3a^2x}{2}\sqrt{X} + \frac{3a^4}{2}\operatorname{Arsinh}\frac{x}{a}\right) + C$
$= \frac{1}{4}\left(x\sqrt{X^3} + \frac{3a^2x}{2}\sqrt{X} + \frac{3a^4}{2}\ln\left(x + \sqrt{X}\right)\right) + C_1$

200 $\int x\sqrt{X^3}\,dx = \frac{1}{5}\sqrt{X^5}$

201 $\int x^2\sqrt{X^3}\,dx = \frac{x\sqrt{X^5}}{6} - \frac{a^2x\sqrt{X^3}}{24} - \frac{a^4x\sqrt{X}}{16} - \frac{a^6}{16}\operatorname{Arsinh}\frac{x}{a} + C$
$= \frac{x\sqrt{X^5}}{6} - \frac{a^2x\sqrt{X^3}}{24} - \frac{a^4x\sqrt{X}}{16} - \frac{a^6}{16}\ln\left(x + \sqrt{X}\right) + C_1$.

202. $\int x^3\sqrt{X^3}\,dx = \frac{\sqrt{X^7}}{7} - \frac{a^2\sqrt{X^5}}{5}$

203. $$\int \frac{\sqrt{X^3}}{x}\,dx = \frac{\sqrt{X^3}}{3} + a^2\sqrt{X} - a^3 \ln \frac{a+\sqrt{X}}{x}$$

204. $$\int \frac{\sqrt{X^3}}{x^2}\,dx = -\frac{\sqrt{X^3}}{x} + \frac{3}{2}x\sqrt{X} + \frac{3}{2}a^2 \operatorname{Arsinh}\frac{x}{a} + C$$
$$= -\frac{\sqrt{X^3}}{x} + \frac{3}{2}x\sqrt{X} + \frac{3}{2}a^2 \ln\left(x+\sqrt{X}\right) + C_1.$$

205 $$\int \frac{\sqrt{X^3}}{x^3}dx = -\frac{\sqrt{X^3}}{2x^2} + \frac{3}{2}\sqrt{X} - \frac{3}{2}a \ln\left(\frac{a+\sqrt{X}}{x}\right).$$

206. $$\int \frac{dx}{\sqrt{X^3}} = \frac{x}{a^2\sqrt{X}}.$$

207 $$\int \frac{x\,dx}{\sqrt{X^3}} = -\frac{1}{\sqrt{X}}.$$

208. $$\int \frac{x^2\,dx}{\sqrt{X^3}} = -\frac{x}{\sqrt{X}} + \operatorname{Arsinh}\frac{x}{a} + C = -\frac{x}{\sqrt{X}} + \ln\left(x+\sqrt{X}\right) + C_1.$$

209. $$\int \frac{x^3\,dx}{\sqrt{X^3}} = \sqrt{X} + \frac{a^2}{\sqrt{X}}.$$

210. $$\int \frac{dx}{x\sqrt{X^3}} = \frac{1}{a^2\sqrt{X}} - \frac{1}{a^3}\ln\frac{a+\sqrt{X}}{x}.$$

211. $$\int \frac{dx}{x^2\sqrt{X^3}} = -\frac{1}{a^4}\left(\frac{\sqrt{X}}{x} + \frac{x}{\sqrt{X}}\right).$$

212. $$\int \frac{dx}{x^3\sqrt{X^3}} = -\frac{1}{2a^2x^2\sqrt{X}} - \frac{3}{2a^4\sqrt{X}} + \frac{3}{2a^5}\ln\frac{a+\sqrt{X}}{x}.$$

21.7.2.7 Integrale mit $\sqrt{x^2 - a^2}$

Bezeichnung: $X = x^2 - a^2$

213. $$\int \sqrt{X}\,dx = \frac{1}{2}\left(x\sqrt{X} - a^2 \operatorname{Arcosh}\frac{x}{a}\right) + C$$
$$= \frac{1}{2}\left[x\sqrt{X} - a^2 \ln\left(x+\sqrt{X}\right)\right] + C_1.$$

214. $$\int x\sqrt{X}\,dx = \frac{1}{3}\sqrt{X^3}.$$

215. $$\int x^2\sqrt{X}\,dx = \frac{x}{4}\sqrt{X^3} + \frac{a^2}{8}\left(x\sqrt{X} - a^2 \operatorname{Arcosh}\frac{x}{a}\right) + C$$
$$= \frac{x}{4}\sqrt{X^3} + \frac{a^2}{8}\left[x\sqrt{X} - a^2 \ln\left(x+\sqrt{X}\right)\right] + C_1$$

216. $$\int x^3\sqrt{X}\,dx = \frac{\sqrt{X^5}}{5} + \frac{a^2\sqrt{X^3}}{3}.$$

217. $$\int \frac{\sqrt{X}}{x}\,dx = \sqrt{X} - a\arccos\frac{a}{x}.$$

218 $\int \frac{\sqrt{X}}{x^2}\,dx = -\frac{\sqrt{X}}{x} + \operatorname{Arcosh}\frac{x}{a} + C = -\frac{\sqrt{X}}{x} + \ln\left(x+\sqrt{X}\right) + C_1.$

219 $\int \frac{\sqrt{X}}{x^3}\,dx = -\frac{\sqrt{X}}{2x^2} + \frac{1}{2a}\arccos\frac{a}{x}.$

220. $\int \frac{dx}{\sqrt{X}} = \operatorname{Arcosh}\frac{x}{a} + C = \ln\left(x+\sqrt{X}\right) + C_1$

221 $\int \frac{x\,dx}{\sqrt{X}} = \sqrt{X}.$

222. $\int \frac{x^2\,dx}{\sqrt{X}} = \frac{x}{2}\sqrt{X} + \frac{a^2}{2}\operatorname{Arcosh}\frac{x}{a} + C = \frac{x}{2}\sqrt{X} + \frac{a^2}{2}\ln\left(x+\sqrt{X}\right) + C_1.$

223. $\int \frac{x^3\,dx}{\sqrt{X}} = \frac{\sqrt{X^3}}{3} + a^2\sqrt{X}$

224 $\int \frac{dx}{x\sqrt{X}} = \frac{1}{a}\arccos\frac{a}{x}$

225. $\int \frac{dx}{x^2\sqrt{X}} = \frac{\sqrt{X}}{a^2 x}.$

226 $\int \frac{dx}{x^3\sqrt{X}} = \frac{\sqrt{X}}{2a^2x^2} + \frac{1}{2a^3}\arccos\frac{a}{x}.$

227 $$\int \sqrt{X^3}\,dx = \frac{1}{4}\left(x\sqrt{X^3} - \frac{3a^2x}{2}\sqrt{X} + \frac{3a^4}{2}\operatorname{Arcosh}\frac{x}{a}\right) + C$$
$$= \frac{1}{4}\left(x\sqrt{X^3} - \frac{3a^2x}{2}\sqrt{X} + \frac{3a^4}{2}\ln\left(x+\sqrt{X}\right)\right) + C_1$$

228. $\int x\sqrt{X^3}\,dx = \frac{1}{5}\sqrt{X^5}$

229 $$\int x^2\sqrt{X^3}\,dx = \frac{x\sqrt{X^5}}{6} + \frac{a^2x\sqrt{X^3}}{24} - \frac{a^4x\sqrt{X}}{16} + \frac{a^6}{16}\operatorname{Arcosh}\frac{x}{a} + C$$
$$= \frac{x\sqrt{X^5}}{6} + \frac{a^2x\sqrt{X^3}}{24} - \frac{a^4x\sqrt{X}}{16} + \frac{a^6}{16}\ln\left(x+\sqrt{X}\right) + C_1.$$

230 $\int x^3\sqrt{X^3}\,dx = \frac{\sqrt{X^7}}{7} + \frac{a^2\sqrt{X^5}}{5}$

231. $\int \frac{\sqrt{X^3}}{x}\,dx = \frac{\sqrt{X^3}}{3} - a^2\sqrt{X} + a^3\arccos\frac{a}{x}.$

232. $$\int \frac{\sqrt{X^3}}{x^2}\,dx = -\frac{\sqrt{X^3}}{2} + \frac{3}{2}x\sqrt{X} - \frac{3}{2}a^2\operatorname{Arcosh}\frac{x}{a} + C$$
$$= -\frac{\sqrt{X^3}}{2} + \frac{3}{2}x\sqrt{X} - \frac{3}{2}a^2\ln\left(x+\sqrt{X}\right) + C_1.$$

233 $\int \frac{\sqrt{X^3}}{x^3}\,dx = -\frac{\sqrt{X^3}}{2x^2} + \frac{3\sqrt{X}}{2} - \frac{3}{2}a\arccos\frac{a}{x}.$

234 $\int \frac{dx}{\sqrt{X^3}} = -\frac{x}{a^2\sqrt{X}}.$

235. $\int \frac{x\,dx}{\sqrt{X^3}} = -\frac{1}{\sqrt{X}}$

236 $\int \frac{x^2\,dx}{\sqrt{X^3}} = -\frac{x}{\sqrt{X}} + \operatorname{Arcosh}\frac{x}{a} + C = -\frac{x}{\sqrt{X}} + \ln\left(x + \sqrt{X}\right) + C_1$

237 $\int \frac{x^3\,dx}{\sqrt{X^3}} = \sqrt{X} - \frac{a^2}{\sqrt{X}}.$

238 $\int \frac{dx}{x\sqrt{X^3}} = -\frac{1}{a^2\sqrt{X}} - \frac{1}{a^3}\arccos\frac{a}{x}$

239 $\int \frac{dx}{x^2\sqrt{X^3}} = -\frac{1}{a^4}\left(\frac{\sqrt{X}}{x} + \frac{x}{\sqrt{X}}\right).$

240. $\int \frac{dx}{x^3\sqrt{X^3}} = \frac{1}{2a^2x^2\sqrt{X}} - \frac{3}{2a^4\sqrt{X}} - \frac{3}{2a^5}\arccos\frac{a}{x}$

21.7.2.8 Integrale mit $\sqrt{ax^2 + bx + c}$

Bezeichnungen: $X = ax^2 + bx + c\,,\ \Delta = 4ac - b^2\,,\ k = \frac{4a}{\Delta}$

241. $$\int \frac{dx}{\sqrt{X}} = \begin{cases} \frac{1}{\sqrt{a}}\ln\left(2\sqrt{aX} + 2ax + b\right) + C & \text{für } a > 0\,, \\ \frac{1}{\sqrt{a}}\operatorname{Arsinh}\frac{2ax + b}{\sqrt{\Delta}} + C_1 & \text{für } a > 0\,,\ \Delta > 0\,, \\ \frac{1}{\sqrt{a}}\ln(2ax + b) & \text{für } a > 0\,,\ \Delta = 0\,, \\ -\frac{1}{\sqrt{-a}}\arcsin\frac{2ax + b}{\sqrt{-\Delta}} & \text{für } a < 0\,,\ \Delta < 0\,. \end{cases}$$

242 $\int \frac{dx}{X\sqrt{X}} = \frac{2(2ax + b)}{\Delta\sqrt{X}}$

243 $\int \frac{dx}{X^2\sqrt{X}} = \frac{2(2ax + b)}{3\Delta\sqrt{X}}\left(\frac{1}{X} + 2k\right)$

244 $\int \frac{dx}{X^{(2n+1)/2}} = \frac{2(2ax + b)}{(2n - 1)\Delta X^{(2n-1)/2}} + \frac{2k(n - 1)}{2n - 1}\int \frac{dx}{X^{(2n-1)/2}}$

245. $\int \sqrt{X}\,dx = \frac{(2ax + b)\sqrt{X}}{4a} + \frac{1}{2k}\int \frac{dx}{\sqrt{X}}$ (s. Nr. 241).

246. $\int X\sqrt{X}\,dx = \frac{(2ax + b)\sqrt{X}}{8a}\left(X + \frac{3}{2k}\right) + \frac{3}{8k^2}\int \frac{dx}{\sqrt{X}}$ (s. Nr 241)

247. $\int X^2\sqrt{X}\,dx = \frac{(2ax + b)\sqrt{X}}{12a}\left(X^2 + \frac{5X}{4k} + \frac{15}{8k^2}\right) + \frac{5}{16k^3}\int \frac{dx}{\sqrt{X}}$ (s. Nr 241)

248. $\int X^{(2n+1)/2}\,dx = \frac{(2ax + b)X^{(2n+1)/2}}{4a(n + 1)} + \frac{2n + 1}{2k(n + 1)}\int X^{(2n-1)/2}\,dx$

249. $\int \frac{x\,dx}{\sqrt{X}} = \frac{\sqrt{X}}{a} - \frac{b}{2a}\int \frac{dx}{\sqrt{X}}$ (s Nr 241).

250. $\int \frac{x\,dx}{X\sqrt{X}} = -\frac{2(bx+2c)}{\Delta\sqrt{X}}$

251. $\int \frac{x\,dx}{X^{(2n+1)/2}} = -\frac{1}{(2n-1)aX^{(2n-1)/2}} - \frac{b}{2a}\int \frac{dx}{X^{(2n+1)/2}}$ (s. Nr. 244).

252 $\int \frac{x^2\,dx}{\sqrt{X}} = \left(\frac{x}{2a} - \frac{3b}{4a^2}\right)\sqrt{X} + \frac{3b^2-4ac}{8a^2}\int \frac{dx}{\sqrt{X}}$ (s. Nr. 241).

253 $\int \frac{x^2\,dx}{X\sqrt{X}} = \frac{(2b^2-4ac)x+2bc}{a\Delta\sqrt{X}} + \frac{1}{a}\int \frac{dx}{\sqrt{X}}$ (s. Nr 241).

254. $\int x\sqrt{X}\,dx = \frac{X\sqrt{X}}{3a} - \frac{b(2ax+b)}{8a^2}\sqrt{X} - \frac{b}{4ak}\int \frac{dx}{\sqrt{X}}$ (s. Nr. 241).

255. $\int xX\sqrt{X}\,dx = \frac{X^2\sqrt{X}}{5a} - \frac{b}{2a}\int X\sqrt{X}\,dx$ (s. Nr 246).

256 $\int xX^{(2n+1)/2}\,dx = \frac{X^{(2n+3)/2}}{(2n+3)a} - \frac{b}{2a}\int X^{(2n+1)/2}\,dx$ (s Nr. 248).

257. $\int x^2\sqrt{X}\,dx = \left(x - \frac{5b}{6a}\right)\frac{X\sqrt{X}}{4a} + \frac{5b^2-4ac}{16a^2}\int \sqrt{X}\,dx$ (s. Nr 245).

258 $$\int \frac{dx}{x\sqrt{X}} = \begin{cases} -\frac{1}{\sqrt{c}}\ln\left(\frac{2\sqrt{cX}}{x} + \frac{2c}{x} + b\right) + C & \text{für } c > 0\,, \\ -\frac{1}{\sqrt{c}}\operatorname{Arsinh}\frac{bx+2c}{x\sqrt{\Delta}} + C_1 & \text{für } c > 0\,,\ \Delta > 0\,, \\ -\frac{1}{\sqrt{c}}\ln\frac{bx+2c}{x} & \text{für } c > 0\,,\ \Delta = 0\,, \\ \frac{1}{\sqrt{-c}}\arcsin\frac{bx+2c}{x\sqrt{-\Delta}} & \text{für } c < \,,\ \Delta < 0\,. \end{cases}$$

259 $\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{cx} - \frac{b}{2c}\int \frac{dx}{x\sqrt{X}}$ (s. Nr. 258)

260 $\int \frac{\sqrt{X}\,dx}{x} = \sqrt{X} + \frac{b}{2}\int \frac{dx}{\sqrt{X}} + c\int \frac{dx}{x\sqrt{X}}$ (s. Nr. 241 und 258).

261. $\int \frac{\sqrt{X}\,dx}{x^2} = -\frac{\sqrt{X}}{x} + a\int \frac{dx}{\sqrt{X}} + \frac{b}{2}\int \frac{dx}{x\sqrt{X}}$ (s. Nr. 241 und 258)

262. $\int \frac{X^{(2n+1)/2}}{x}dx = \frac{X^{(2n+1)/2}}{2n+1} + \frac{b}{2}\int X^{(2n-1)/2}\,dx + c\int \frac{X^{(2n-1)/2}}{x}dx$ (s Nr. 248 und 260)

263. $\int \frac{dx}{x\sqrt{ax^2+bx}} = -\frac{2}{bx}\sqrt{ax^2+bx}.$

264. $\int \frac{dx}{\sqrt{2ax-x^2}} = \arcsin\frac{x-a}{a}.$

265. $\int \frac{x\,dx}{\sqrt{2ax-x^2}} = -\sqrt{2ax-x^2} + a\arcsin\frac{x-a}{a}$

266 $\int \sqrt{2ax-x^2}\,dx = \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\arcsin\frac{x-a}{a}$

267 $\int \frac{dx}{(ax^2+b)\sqrt{fx^2+g}} = \frac{1}{\sqrt{b}\sqrt{ag-bf}}\arctan\frac{x\sqrt{ag-bf}}{\sqrt{b}\sqrt{fx^2+g}}$ $(ag-bf>0)$,

$$= \frac{1}{2\sqrt{b}\sqrt{bf-ag}}\ln\frac{\sqrt{b}\sqrt{fx^2+g}+x\sqrt{bf-ag}}{\sqrt{b}\sqrt{fx^2+g}-x\sqrt{bf-ag}} \qquad (ag-bf<0)$$

21.7.2.9 Integrale mit anderen irrationalen Ausdrücken

268. $\int \sqrt[n]{ax+b}\,dx = \frac{n(ax+b)}{(n+1)a}\sqrt[n]{ax+b}.$

269 $\int \frac{dx}{\sqrt[n]{ax+b}} = \frac{n(ax+b)}{(n-1)a}\frac{1}{\sqrt[n]{ax+b}}.$

270 $\int \frac{dx}{x\sqrt{x^n+a^2}} = -\frac{2}{na}\ln\frac{a+\sqrt{x^n+a^2}}{\sqrt{x^n}}.$

271. $\int \frac{dx}{x\sqrt{x^n-a^2}} = \frac{2}{na}\arccos\frac{a}{\sqrt{x^n}}.$

272. $\int \frac{\sqrt{x}\,dx}{\sqrt{a^3-x^3}} = \frac{2}{3}\arcsin\sqrt{\left(\frac{x}{a}\right)^3}.$

21.7.2.10 Rekursionsformeln für ein Integral mit binomischem Differential

273. $\int x^m(ax^n+b)^p\,dx$

$$= \frac{1}{m+np+1}\left[x^{m+1}(ax^n+b)^p + npb\int x^m(ax^n+b)^{p-1}\,dx\right],$$

$$= \frac{1}{bn(p+1)}\left[-x^{m+1}(ax^n+b)^{p+1} + (m+n+np+1)\int x^m(ax^n+b)^{p+1}\,dx\right],$$

$$= \frac{1}{(m+1)b}\left[x^{m+1}(ax^n+b)^{p+1} - a(m+n+np+1)\int x^{m+n}(ax^n+b)^p\,dx\right],$$

$$= \frac{1}{a(m+np+1)}\left[x^{m-n+1}(ax^n+b)^{p+1} - (m-n+1)b\int x^{m-n}(ax^n+b)^p\,dx\right]$$

21.7.3 Integrale trigonometrischer Funktionen

(Integrale von Funktionen, die neben Hyperbel- und Exponentialfunktionen auch die Funktionen $\sin x$ und $\cos x$ enthalten sind in den Tabellen Integrale anderer transzendenter Funktionen (s. 21 7 4, S 1080 aufgeführt)

21.7.3.1 Integrale mit Sinusfunktion

274 $\int \sin ax\,dx = -\frac{1}{a}\cos ax$

275 $\int \sin^2 ax\,dx = \frac{1}{2}x - \frac{1}{4a}\sin 2ax.$

276. $\int \sin^3 ax\, dx = -\frac{1}{a}\cos ax + \frac{1}{3a}\cos^3 ax$

277 $\int \sin^4 ax\, dx = \frac{3}{8}x - \frac{1}{4a}\sin 2ax + \frac{1}{32a}\sin 4ax$

278. $\int \sin^n ax\, dx = -\frac{\sin^{n-1} ax \cos ax}{na} + \frac{n-1}{n}\int \sin^{n-2} ax\, dx$ (n ganzzahlig, > 0)

279. $\int x \sin ax\, dx = \frac{\sin ax}{a^2} - \frac{x\cos ax}{a}.$

280 $\int x^2 \sin ax\, dx = \frac{2x}{a^2}\sin ax - \left(\frac{x^2}{a} - \frac{2}{a^3}\right)\cos ax$

281. $\int x^3 \sin ax\, dx = \left(\frac{3x^2}{a^2} - \frac{6}{a^4}\right)\sin ax - \left(\frac{x^3}{a} - \frac{6x}{a^3}\right)\cos ax.$

282. $\int x^n \sin ax\, dx = -\frac{x^n}{a}\cos ax + \frac{n}{a}\int x^{n-1}\cos ax\, dx$ ($n > 0$).

283. $\int \frac{\sin ax}{x}dx = ax - \frac{(ax)^3}{3\cdot 3!} + \frac{(ax)^5}{5\cdot 5!} - \frac{(ax)^7}{7\cdot 7!} + \cdots .$

Das bestimmte Integral $\int\limits_0^x \frac{\sin t}{t}\,dt$ nennt man Integralsinus (s. 8.2.5,**1.**, S. 477) und bezeichnet es mit si(x)

Die Berechnung des Integrals s. 14 4 3.2,**2.**, S. 719. Die Reihenentwicklung $\mathrm{si}(x) = x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \frac{x^7}{7\cdot 7!} + \cdots$ s 8 2 5,**1.**, S. 477.

284 $\int \frac{\sin ax}{x^2}dx = -\frac{\sin ax}{x} + a\int \frac{\cos ax\, dx}{x}$ (s. Nr.322).

285 $\int \frac{\sin ax}{x^n}dx = -\frac{1}{n-1}\frac{\sin ax}{x^{n-1}} + \frac{a}{n-1}\int \frac{\cos ax}{x^{n-1}}\,dx$ (s. Nr.324).

286. $\int \frac{dx}{\sin ax} = \int \operatorname{cosec} ax\, dx = \frac{1}{a}\ln\tan\frac{ax}{2} = \frac{1}{a}\ln(\operatorname{cosec} ax \cot ax).$

287. $\int \frac{dx}{\sin^2 ax} = -\frac{1}{a}\cot ax.$

288 $\int \frac{dx}{\sin^3 ax} = -\frac{\cos ax}{2a\sin^2 ax} + \frac{1}{2a}\ln\tan\frac{ax}{2}.$

289. $\int \frac{dx}{\sin^n ax} = -\frac{1}{a(n-1)}\frac{\cos ax}{\sin^{n-1} ax} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2} ax}$ ($n > 1$).

290. $\int \frac{x\,dx}{\sin ax} = \frac{1}{a^2}\left(ax + \frac{(ax)^3}{3\cdot 3!} + \frac{7(ax)^5}{3\cdot 5\cdot 5!} + \frac{31(ax)^7}{3\cdot 7\cdot 7!}\right.$

$$+\frac{127(ax)^9}{3\cdot 5\cdot 9!}+\cdots+\frac{2(2^{2n-1}-1)}{(2n+1)!}B_n(ax)^{2n+1}+\cdots\Big)$$

Mit B_n sind die BERNOULLIschen Zahlen (s 7.2 4.2, S 428) bezeichnet.

291 $$\int\frac{x\,dx}{\sin^2 ax}=-\frac{x}{a}\cot ax+\frac{1}{a^2}\ln\sin ax$$

292 $$\int\frac{x\,dx}{\sin^n ax}=-\frac{x\cos ax}{(n-1)a\sin^{n-1}ax}-\frac{1}{(n-1)(n-2)a^2\sin^{n-2}ax}+\frac{n-2}{n-1}\int\frac{x\,dx}{\sin^{n-2}ax}\qquad(n>2)$$

293 $$\int\frac{dx}{1+\sin ax}=-\frac{1}{a}\tan\left(\frac{\pi}{4}-\frac{ax}{2}\right).$$

294 $$\int\frac{dx}{1-\sin ax}=\frac{1}{a}\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)$$

295 $$\int\frac{x\,dx}{1+\sin ax}=-\frac{x}{a}\tan\left(\frac{\pi}{4}-\frac{ax}{2}\right)+\frac{2}{a^2}\ln\cos\left(\frac{\pi}{4}-\frac{ax}{2}\right).$$

196 $$\int\frac{x\,dx}{1-\sin ax}=\frac{x}{a}\cot\left(\frac{\pi}{4}-\frac{ax}{2}\right)+\frac{2}{a^2}\ln\sin\left(\frac{\pi}{4}-\frac{ax}{2}\right).$$

297. $$\int\frac{\sin ax\,dx}{1\pm\sin ax}=\pm x+\frac{1}{a}\tan\left(\frac{\pi}{4}\mp\frac{ax}{2}\right).$$

298. $$\int\frac{dx}{\sin ax(1\pm\sin ax)}=\frac{1}{a}\tan\left(\frac{\pi}{4}\mp\frac{ax}{2}\right)+\frac{1}{a}\ln\tan\frac{ax}{2}$$

299 $$\int\frac{dx}{(1+\sin ax)^2}=-\frac{1}{2a}\tan\left(\frac{\pi}{4}-\frac{ax}{2}\right)-\frac{1}{6a}\tan^3\left(\frac{\pi}{4}-\frac{ax}{2}\right).$$

300. $$\int\frac{dx}{(1-\sin ax)^2}=\frac{1}{2a}\cot\left(\frac{\pi}{4}-\frac{ax}{2}\right)+\frac{1}{6a}\cot^3\left(\frac{\pi}{4}-\frac{ax}{2}\right)$$

301 $$\int\frac{\sin ax\,dx}{(1+\sin ax)^2}=-\frac{1}{2a}\tan\left(\frac{\pi}{4}-\frac{ax}{2}\right)+\frac{1}{6a}\tan^3\left(\frac{\pi}{4}-\frac{ax}{2}\right).$$

302. $$\int\frac{\sin ax\,dx}{(1-\sin ax)^2}=-\frac{1}{2a}\cot\left(\frac{\pi}{4}-\frac{ax}{2}\right)+\frac{1}{6a}\cot^3\left(\frac{\pi}{4}-\frac{ax}{2}\right)$$

303 $$\int\frac{dx}{1+\sin^2 ax}=\frac{1}{2\sqrt{2}a}\arcsin\left(\frac{3\sin^2 ax-1}{\sin^2 ax+1}\right).$$

304. $$\int\frac{dx}{1-\sin^2 ax}=\int\frac{dx}{\cos^2 ax}=\frac{1}{a}\tan ax$$

305 $$\int\sin ax\sin bx\,dx=\frac{\sin(a-b)x}{2(a-b)}-\frac{\sin(a+b)x}{2(a+b)}\qquad(|a|\neq|b|,\qquad\text{für } |a|=|b| \text{ s Nr } 275)$$

306 $$\int\frac{dx}{b+c\sin ax}=\frac{2}{a\sqrt{b^2-c^2}}\arctan\frac{b\tan ax/2+c}{\sqrt{b^2-c^2}}\qquad\text{für } b^2>c^2,$$
$$=\frac{1}{a\sqrt{c^2-b^2}}\ln\frac{b\tan ax/2+c-\sqrt{c^2-b^2}}{b\tan ax/2+c+\sqrt{c^2-b^2}}\qquad\text{für } b^2<c^2.$$

307 $\int \frac{\sin ax\,dx}{b + c\sin ax} = \frac{x}{c} - \frac{b}{c}\int \frac{dx}{b + c\sin ax}$ (s Nr 306)

308. $\int \frac{dx}{\sin ax(b + c\sin ax)} = \frac{1}{ab}\ln\tan\frac{ax}{2} - \frac{c}{b}\int \frac{dx}{b + c\sin ax}$ (s Nr 306)

309 $\int \frac{dx}{(b + c\sin ax)^2} = \frac{c\cos ax}{a(b^2 - c^2)(b + c\sin ax)} + \frac{b}{b^2 - c^2}\int \frac{dx}{b + c\sin ax}$ (s. Nr.306).

310 $\int \frac{\sin ax\,dx}{(b + c\sin ax)^2} = \frac{b\cos ax}{a(c^2 - b^2)(b + c\sin ax)} + \frac{c}{c^2 - b^2}\int \frac{dx}{b + c\sin ax}$ (s Nr 306).

311 $\int \frac{dx}{b^2 + c^2\sin^2 ax} = \frac{1}{ab\sqrt{b^2 + c^2}}\arctan\frac{\sqrt{b^2 + c^2}\tan ax}{b}$ $(b > 0)$.

312 $\int \frac{dx}{b^2 - c^2\sin^2 ax} = \frac{1}{ab\sqrt{b^2 - c^2}}\arctan\frac{\sqrt{b^2 - c^2}\tan ax}{b}$ $(b^2 > c^2,\ b > 0)$,

$= \frac{1}{2ab\sqrt{c^2 - b^2}}\ln\frac{\sqrt{c^2 - b^2}\tan ax + b}{\sqrt{c^2 - b^2}\tan ax - b}$ $(c^2 > b^2,\ b > 0)$

21.7.3.2 Integrale mit Kosinusfunktion

313. $\int \cos ax\,dx = \frac{1}{a}\sin ax$

314. $\int \cos^2 ax\,dx = \frac{1}{2}x + \frac{1}{4a}\sin 2ax.$

315 $\int \cos^3 ax\,dx = \frac{1}{a}\sin ax - \frac{1}{3a}\sin^3 ax.$

316 $\int \cos^4 ax\,dx = \frac{3}{8}x + \frac{1}{4a}\sin 2ax + \frac{1}{32a}\sin 4ax.$

317. $\int \cos^n ax\,dx = \frac{\cos^{n-1} ax\sin ax}{na} + \frac{n-1}{n}\int \cos^{n-2} ax\,dx.$

318 $\int x\cos ax\,dx = \frac{\cos ax}{a^2} + \frac{x\sin ax}{a}$

319 $\int x^2\cos ax\,dx = \frac{2x}{a^2}\cos ax + \left(\frac{x^2}{a} - \frac{2}{a^3}\right)\sin ax.$

320. $\int x^3\cos ax\,dx = \left(\frac{3x^2}{a^2} - \frac{6}{a^4}\right)\cos ax + \left(\frac{x^3}{a} - \frac{6x}{a^3}\right)\sin ax.$

321. $\int x^n\cos ax\,dx = \frac{x^n\sin ax}{a} - \frac{n}{a}\int x^{n-1}\sin ax\,dx.$

322. $\int \frac{\cos ax}{x}dx = \ln(ax) - \frac{(ax)^2}{2\cdot 2!} + \frac{(ax)^4}{4\cdot 4!} - \frac{(ax)^6}{6\cdot 6!} + \cdots$

Das bestimmte Integral $-\int\limits_x^\infty \frac{\cos t}{t}\,dt$ nennt man Integralkosinus und bezeichnet es mit $\mathrm{Ci}(x)$ Es gilt die Reihenentwicklung $\mathrm{Ci}(x) = C + \ln x - \frac{x^2}{2\cdot 2!} + \frac{x^4}{4\cdot 4!} - \frac{x^6}{6\cdot 6!} + \cdots$ (s. 8.2 5,**2.**, S 477); mit C ist die EULERsche Konstante (s 8 2 5,**2.**, S 477)) bezeichnet

323 $\int \frac{\cos ax}{x^2} dx = -\frac{\cos ax}{x} - a \int \frac{\sin ax\, dx}{x}$ (s Nr 283)

324. $\int \frac{\cos ax}{x^n} dx = -\frac{\cos ax}{(n-1)x^{n-1}} - \frac{a}{n-1} \int \frac{\sin ax\, dx}{x^{n-1}}$ $(n \neq 1)$(s Nr.285).

325 $\int \frac{dx}{\cos ax} = \frac{1}{a}$ Artanh Artanh$(\sin ax) = \frac{1}{a} \ln \tan\left(\frac{ax}{2} + \frac{\pi}{4}\right) = \frac{1}{a} \ln(\sec ax + \tan ax)$

326 $\int \frac{dx}{\cos^2 ax} = \frac{1}{a} \tan ax$

327 $\int \frac{dx}{\cos^3 ax} = \frac{\sin ax}{2a \cos^2 ax} + \frac{1}{2a} \ln \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$

328 $\int \frac{dx}{\cos^n ax} = \frac{1}{a(n-1)} \frac{\sin ax}{\cos^{n-1} ax} + \frac{n-2}{n-1} \int \frac{dx}{\cos^{n-2} ax}$ $(n > 1)$.

329 $\int \frac{x\, dx}{\cos ax} = \frac{1}{a^2} \left(\frac{(ax)^2}{2} + \frac{(ax)^4}{4 \cdot 2!} + \frac{5(ax)^6}{6 \cdot 4!} + \frac{61(ax)^8}{8 \cdot 6!} + \frac{1385(ax)^{10}}{10\ 8!} + \cdots \right.$

$\left. + \frac{E_n (ax)^{2n+2}}{(2n+2)(2n!)} + \cdots \right)$

Mit E_n sind die EULERschen Zahlen (s 7 2 4 2. S 429) bezeichnet

330. $\int \frac{x\, dx}{\cos^2 ax} = \frac{x}{a} \tan ax + \frac{1}{a^2} \ln \cos ax$

331. $\int \frac{x\, dx}{\cos^n ax} = \frac{x \sin ax}{(n-1) a \cos^{n-1} ax} - \frac{1}{(n-1)(n-2) a^2 \cos^{n-2} ax} + \frac{n-2}{n-1} \int \frac{x\, dx}{\cos^{n-2} ax}$ $(n > 2)$

332 $\int \frac{dx}{1 + \cos ax} = \frac{1}{a} \tan \frac{ax}{2}.$

333 $\int \frac{dx}{1 - \cos ax} = -\frac{1}{a} \cot \frac{ax}{2}$

334. $\int \frac{x\, dx}{1 + \cos ax} = \frac{x}{a} \tan \frac{ax}{2} + \frac{2}{a^2} \ln \cos \frac{ax}{2}$

335 $\int \frac{x\, dx}{1 - \cos ax} = -\frac{x}{a} \cot \frac{ax}{2} + \frac{2}{a^2} \ln \sin \frac{ax}{2}$

336. $\int \frac{\cos ax\, dx}{1 + \cos ax} = x - \frac{1}{a} \tan \frac{ax}{2}.$

337 $\int \frac{\cos ax\, dx}{1 - \cos ax} = -x - \frac{1}{a} \cot \frac{ax}{2}$

338. $\int \frac{dx}{\cos ax(1 + \cos ax)} = \frac{1}{a} \ln \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right) - \frac{1}{a} \tan \frac{ax}{2}$

339 $\int \frac{dx}{\cos ax(1 - \cos ax)} = \frac{1}{a} \ln \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right) - \frac{1}{a} \cot \frac{ax}{2}$

340 $\int \frac{dx}{(1+\cos ax)^2} = \frac{1}{2a}\tan\frac{ax}{2} + \frac{1}{6a}\tan^3\frac{ax}{2}.$

341 $\int \frac{dx}{(1-\cos ax)^2} = -\frac{1}{2a}\cot\frac{ax}{2} - \frac{1}{6a}\cot^3\frac{ax}{2}.$

342 $\int \frac{\cos ax\,dx}{(1+\cos ax)^2} = \frac{1}{2a}\tan\frac{ax}{2} - \frac{1}{6a}\tan^3\frac{ax}{2}.$

343. $\int \frac{\cos ax\,dx}{(1-\cos ax)^2} = \frac{1}{2a}\cot\frac{ax}{2} - \frac{1}{6a}\cot^3\frac{ax}{2}$

344 $\int \frac{dx}{1+\cos^2 ax} = \frac{1}{2\sqrt{2}a}\arcsin\left(\frac{1-3\cos^2 ax}{1+\cos^2 ax}\right)$

345 $\int \frac{dx}{1-\cos^2 ax} = \int \frac{dx}{\sin^2 ax} = -\frac{1}{a}\cot ax$

346 $\int \cos ax \cos bx\,dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)}$ $(|a| \neq |b|)$; (für $|a| = |b|$ s Nr 314)

347 $\int \frac{dx}{b+c\cos ax} = \frac{2}{a\sqrt{b^2-c^2}}\arctan\frac{(b-c)\tan ax/2}{\sqrt{b^2-c^2}}$ (für $b^2 > c^2$)

$= \frac{1}{a\sqrt{c^2-b^2}}\ln\frac{(c-b)\tan ax/2 + \sqrt{c^2-b^2}}{(c-b)\tan ax/2 - \sqrt{c^2-b^2}}$ (für $b^2 < c^2$)

348 $\int \frac{\cos ax\,dx}{b+c\cos ax} = \frac{x}{c} - \frac{b}{c}\int \frac{dx}{b+c\cos ax}$ (s Nr.347)

349 $\int \frac{dx}{\cos ax(b+c\cos ax)} = \frac{1}{ab}\ln\tan\left(\frac{ax}{2}+\frac{\pi}{4}\right) - \frac{c}{b}\int \frac{dx}{b+c\cos ax}$ (s. Nr 347)

350 $\int \frac{dx}{(b+c\cos ax)^2} = \frac{c\sin ax}{a(c^2-b^2)(b+c\cos ax)} - \frac{b}{c^2-b^2}\int \frac{dx}{b+c\cos ax}$ (s Nr.347)

351 $\int \frac{\cos ax\,dx}{(b+c\cos ax)^2} = \frac{b\sin ax}{a(b^2-c^2)(b+c\cos ax)} - \frac{c}{b^2-c^2}\int \frac{dx}{b+c\cos ax}$ (s Nr 347)

352 $\int \frac{dx}{b^2+c^2\cos^2 ax} = \frac{1}{ab\sqrt{b^2+c^2}}\arctan\frac{b\tan ax}{\sqrt{b^2+c^2}}$ $(b > 0)$

353. $\int \frac{dx}{b^2-c^2\cos^2 ax} = \frac{1}{ab\sqrt{b^2-c^2}}\arctan\frac{b\tan ax}{\sqrt{b^2-c^2}}$ $(b^2 > c^2,\ b > 0)$.

$= \frac{1}{2ab\sqrt{c^2-b^2}}\ln\frac{b\tan ax - \sqrt{c^2-b^2}}{b\tan ax + \sqrt{c^2-b^2}}$ $(c^2 > b^2,\ b > 0)$

21.7.3.3 Integrale mit Sinus- und Kosinusfunktion

354 $\int \sin ax \cos ax\,dx = \frac{1}{2a}\sin^2 ax$

355 $\int \sin^2 ax \cos^2 ax\,dx = \frac{x}{8} - \frac{\sin 4ax}{32a}.$

356. $\int \sin^n ax \cos ax\,dx = \frac{1}{a(n+1)}\sin^{n+1} ax$ $(n \neq -1)$.

357. $\int \sin ax \cos^n ax \, dx = -\frac{1}{a(n+1)} \cos^{n+1} ax \qquad (n \neq -1).$

358 $\int \sin^n ax \cos^m ax \, dx = -\frac{\sin^{n-1} ax \cos^{m+1} ax}{a(n+m)} + \frac{n-1}{n+m} \int \sin^{n-2} ax \cos^m ax \, dx$

(Erniedrigung der Potenz n, m und $n > 0$),

$= \frac{\sin^{n+1} ax \cos^{m-1} ax}{a(n+m)} + \frac{m-1}{n+m} \int \sin^n ax \cos^{m-2} ax \, dx$

(Erniedrigung der Potenz m; m und $n > 0$)

359. $\int \frac{dx}{\sin ax \cos ax} = \frac{1}{a} \ln \tan ax.$

360 $\int \frac{dx}{\sin^2 ax \cos ax} = \frac{1}{a} \left[\ln \tan \left(\frac{\pi}{4} + \frac{ax}{2} \right) - \frac{1}{\sin ax} \right]$

361. $\int \frac{dx}{\sin ax \cos^2 ax} = \frac{1}{a} \left(\ln \tan \frac{ax}{2} + \frac{1}{\cos ax} \right)$

362. $\int \frac{dx}{\sin^3 ax \cos ax} = \frac{1}{a} \left(\ln \tan ax - \frac{1}{2 \sin^2 ax} \right).$

363. $\int \frac{dx}{\sin ax \cos^3 ax} = \frac{1}{a} \left(\ln \tan ax + \frac{1}{2 \cos^2 ax} \right)$

364. $\int \frac{dx}{\sin^2 ax \cos^2 ax} = -\frac{2}{a} \cot 2ax$

365. $\int \frac{dx}{\sin^2 ax \cos^3 ax} = \frac{1}{a} \left[\frac{\sin ax}{2 \cos^2 ax} - \frac{1}{\sin ax} + \frac{3}{2} \ln \tan \left(\frac{\pi}{4} + \frac{ax}{2} \right) \right]$

366. $\int \frac{dx}{\sin^3 ax \cos^2 ax} = \frac{1}{a} \left(\frac{1}{\cos ax} - \frac{\cos ax}{2 \sin^2 ax} + \frac{3}{2} \ln \tan \frac{ax}{2} \right).$

367 $\int \frac{dx}{\sin ax \cos^n ax} = \frac{1}{a(n-1) \cos^{n-1} ax} + \int \frac{dx}{\sin ax \cos^{n-2} ax} \qquad (n \neq 1)$ (s Nr 361 und 363)

368. $\int \frac{dx}{\sin^n ax \cos ax} = -\frac{1}{a(n-1) \sin^{n-1} ax} + \int \frac{dx}{\sin^{n-2} ax \cos ax} \qquad (n \neq 1)$(s Nr.360 und 362)

369. $\int \frac{dx}{\sin^n ax \cos^m ax} = -\frac{1}{a(n-1)} \cdot \frac{1}{\sin^{n-1} ax \cos^{m-1} ax} + \frac{n+m-2}{n-1} \int \frac{dx}{\sin^{n-2} ax \cos^m ax}$

(Erniedrigung der Potenz n; $m > 0$, $n > 1$),

$= \frac{1}{a(m-1)} \cdot \frac{1}{\sin^{n-1} ax \cos^{m-1} ax} + \frac{n+m-2}{n-1} \int \frac{dx}{\sin^n ax \cos^{m-2} ax}$

(Erniedrigung der Potenz m; $n > 0$, $m > 1$)

370. $\int \frac{\sin ax \, dx}{\cos^2 ax} = \frac{1}{a \cos ax} = \frac{1}{a} \sec ax.$

371 $\int \frac{\sin ax\, dx}{\cos^3 ax} = \frac{1}{2a\cos^2 ax} + C = \frac{1}{2a}\tan^2 ax + C_1.$

372 $\int \frac{\sin ax\, dx}{\cos^n ax} = \frac{1}{a(n-1)\cos^{n-1} ax}$

373 $\int \frac{\sin^2 ax\, dx}{\cos ax} = -\frac{1}{a}\sin ax + \frac{1}{a}\ln\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$

374 $\int \frac{\sin^2 ax\, dx}{\cos^3 ax} = \frac{1}{a}\left[\frac{\sin ax}{2\cos^2 ax} - \frac{1}{2}\ln\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)\right]$

375 $\int \frac{\sin^2 ax\, dx}{\cos^n ax} = \frac{\sin ax}{a(n-1)\cos^{n-1} ax} - \frac{1}{n-1}\int \frac{dx}{\cos^{n-2} ax}$ $(n \neq 1)$ (s. Nr 325, 326, 328)

376 $\int \frac{\sin^3 ax\, dx}{\cos ax} = -\frac{1}{a}\left(\frac{\sin^2 ax}{2} + \ln\cos ax\right)$

377 $\int \frac{\sin^3 ax\, dx}{\cos^2 ax} = \frac{1}{a}\left(\cos ax + \frac{1}{\cos ax}\right)$

378 $\int \frac{\sin^3 ax\, dx}{\cos^n ax} = \frac{1}{a}\left[\frac{1}{(n-1)\cos^{n-1} ax} - \frac{1}{(n-3)\cos^{n-3} ax}\right]$ $(n \neq 1,\ n \neq 3)$

379 $\int \frac{\sin^n ax}{\cos ax}\, dx = -\frac{\sin^{n-1} ax}{a(n-1)} + \int \frac{\sin^{n-2} ax\, dx}{\cos ax}$ $(n \neq 1)$.

380 $\int \frac{\sin^n ax}{\cos^m ax} dx = = \frac{\sin^{n+1} ax}{a(m-1)\cos^{m-1} ax} - \frac{n-m+2}{m-1}\int \frac{\sin^n ax}{\cos^{m-2} ax} dx$ $(m \neq 1)$,

$= -\frac{\sin^{n-1} ax}{a(n-m)\cos^{m-1} ax} + \frac{n-1}{n-m}\int \frac{\sin^{n-2} ax\, dx}{\cos^m ax}$ $(m \neq n)$,

$= \frac{\sin^{n-1} ax}{a(m-1)\cos^{m-1} ax} - \frac{n-1}{m-1}\int \frac{\sin^{n-1} ax\, dx}{\cos^{m-2} ax}$ $(m \neq 1)$.

381 $\int \frac{\cos ax\, dx}{\sin^2 ax} = -\frac{1}{a\sin ax} = -\frac{1}{a}\operatorname{cosec} ax$

382 $\int \frac{\cos ax\, dx}{\sin^3 ax} = -\frac{1}{2a\sin^2 ax} + C = -\frac{\cot^2 ax}{2a} + C_1$

383 $\int \frac{\cos ax\, dx}{\sin^n ax} = -\frac{1}{a(n-1)\sin^{n-1} ax}$

384. $\int \frac{\cos^2 ax\, dx}{\sin ax} = \frac{1}{a}\left(\cos ax + \ln\tan\frac{ax}{2}\right)$

385 $\int \frac{\cos^2 ax\, dx}{\sin^3 ax} = -\frac{1}{2a}\left(\frac{\cos ax}{\sin^2 ax} - \ln\tan\frac{ax}{2}\right).$

386 $\int \frac{\cos^2 ax\, dx}{\sin^n ax} = -\frac{1}{(n-1)}\left(\frac{\cos ax}{a\sin^{n-1} ax} + \int \frac{dx}{\sin^{n-2} ax}\right)$ $(n \neq 1)$ (s Nr 289)

387 $\int \frac{\cos^3 ax\, dx}{\sin ax} = \frac{1}{a}\left(\frac{\cos^2 ax}{2} + \ln\sin ax\right)$

388. $\int \frac{\cos^3 ax\, dx}{\sin^2 ax} = -\frac{1}{a}\left(\sin ax + \frac{1}{\sin ax}\right).$

389. $\int \frac{\cos^3 ax\, dx}{\sin^n ax} = \frac{1}{a}\left[\frac{1}{(n-3)\sin^{n-3} ax} - \frac{1}{(n-1)\sin^{n-1} ax}\right] \quad (n \neq 1, n \neq 3)$

390 $\int \frac{\cos^n ax}{\sin ax}\, dx = \frac{\cos^{n-1} ax}{a(n-1)} + \int \frac{\cos^{n-2} ax\, dx}{\sin ax} (n \neq 1)$

391 $$\int \frac{\cos^n ax\, dx}{\sin^m ax} = -\frac{\cos^{n+1} ax}{a(m-1)\sin^{m-1} ax} - \frac{n-m+2}{m-1}\int \frac{\cos^n ax\, dx}{\sin^{m-2} ax} \quad (m \neq 1),$$
$$= \frac{\cos^{n-1} ax}{a(n-m)\sin^{m-1} ax} + \frac{n-1}{m-1}\int \frac{\cos^{n-2} ax\, dx}{\sin^m ax} \quad (m \neq n),$$
$$= -\frac{\cos^{n-1} ax}{a(m-1)\sin^{m-1} ax} - \frac{n-1}{m-1}\int \frac{\cos^{n-2} ax\, dx}{\sin^{m-2} ax} \quad (m \neq 1).$$

392 $\int \frac{dx}{\sin ax(1 \pm \cos ax)} = \pm\frac{1}{2a(1 \pm \cos ax)} + \frac{1}{2a}\ln\tan\frac{ax}{2}$

393. $\int \frac{dx}{\cos ax(1 \pm \sin ax)} = \mp\frac{1}{2a(1 \pm \sin ax)} + \frac{1}{2a}\ln\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right).$

394 $\int \frac{\sin ax\, dx}{\cos ax(1 \pm \cos ax)} = \frac{1}{a}\ln\frac{1 \pm \cos ax}{\cos ax}$

395. $\int \frac{\cos ax\, dx}{\sin ax(1 \pm \sin ax)} = -\frac{1}{a}\ln\frac{1 \pm \sin ax}{\sin ax}.$

396. $\int \frac{\sin ax\, dx}{\cos ax(1 \pm \sin ax)} = \frac{1}{2a(1 \pm \sin ax)} \pm \frac{1}{2a}\ln\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right).$

397. $\int \frac{\cos ax\, dx}{\sin ax(1 \pm \cos ax)} = -\frac{1}{2a(1 \pm \cos ax)} \pm \frac{1}{2a}\ln\tan\frac{ax}{2}$

398. $\int \frac{\sin ax\, dx}{\sin ax \pm \cos ax} = \frac{x}{2} \mp \frac{1}{2a}\ln(\sin ax \pm \cos ax).$

399 $\int \frac{\cos ax\, dx}{\sin ax \pm \cos ax} = \pm\frac{x}{2} + \frac{1}{2a}\ln(\sin ax \pm \cos ax)$

400. $\int \frac{dx}{\sin ax \pm \cos ax} = \frac{1}{a\sqrt{2}}\ln\tan\left(\frac{ax}{2} \pm \frac{\pi}{8}\right)$

401 $\int \frac{dx}{1 + \cos ax \pm \sin ax} = \pm\frac{1}{a}\ln\left(1 \pm \tan\frac{ax}{2}\right)$

402. $\int \frac{dx}{b\sin ax + c\cos ax} = \frac{1}{a\sqrt{b^2+c^2}}\ln\tan\frac{ax+\theta}{2}$ mit $\sin\theta = \frac{c}{\sqrt{b^2+c^2}}$ und $\tan\theta = \frac{c}{b}$

403. $\int \frac{\sin ax\, dx}{b + c\cos ax} = -\frac{1}{ac}\ln(b + c\cos ax).$

404. $\int \frac{\cos ax\, dx}{b + c\sin ax} = \frac{1}{ac}\ln(b + c\sin ax).$

405 $$\int \frac{dx}{b + c\cos ax + f\sin ax} = \int \frac{d\left(x + \frac{\theta}{a}\right)}{b + \sqrt{c^2 + f^2}\sin(ax + \theta)}$$

mit $\sin\theta = \frac{c}{\sqrt{c^2 + f^2}}$ und $\tan\theta = \frac{c}{f}$ (s Nr 306)

406. $$\int \frac{dx}{b^2\cos^2 ax + c^2\sin^2 ax} = \frac{1}{abc}\arctan\left(\frac{c}{b}\tan ax\right)$$

407 $$\int \frac{dx}{b^2\cos^2 ax - c^2\sin^2 ax} = \frac{1}{2abc}\ln\frac{c\tan ax + b}{c\tan ax - b}$$

408 $$\int \sin ax\cos bx\, dx = -\frac{\cos(a+b)x}{2(a+b)} - \frac{\cos(a-b)x}{2(a-b)} \quad (a^2 \neq b^2);$$ (für $a = b$ s Nr 354)

21.7.3.4 Integrale mit Tangensfunktion

409 $$\int \tan ax\, dx = -\frac{1}{a}\ln\cos ax$$

410. $$\int \tan^2 ax\, dx = \frac{\tan ax}{a} - x.$$

411 $$\int \tan^3 ax\, dx = \frac{1}{2a}\tan^2 ax + \frac{1}{a}\ln\cos ax$$

412 $$\int \tan^n ax\, dx = \frac{1}{a(n-1)}\tan^{n-1} ax - \int \tan^{n-2} ax\, dx$$

413. $$\int x\tan ax\, dx = \frac{ax^3}{3} + \frac{a^3x^5}{15} + \frac{2a^5x^7}{105} + \frac{17a^7x^9}{2835} + \cdots + \frac{2^{2n}(2^{2n}-1)B_n a^{2n-1}x^{2n+1}}{(2n+1)!} + \cdots$$

Mit B_n sind die BERNOULLIschen Zahlen (s. 7.2 4.2, S 428) bezeichnet.

414 $$\int \frac{\tan ax\, dx}{x} = ax + \frac{(ax)^3}{9} + \frac{2(ax)^5}{75} + \frac{17(ax)^7}{2205} + \cdots + \frac{2^{2n}(2^{2n}-1)B_n(ax)^{2n-1}}{(2n-1)(2n!)} + \cdots$$

415 $$\int \frac{\tan^n ax}{\cos^2 ax}\, dx = \frac{1}{a(n+1)}\tan^{n+1} ax \qquad (n \neq -1).$$

416 $$\int \frac{dx}{\tan ax \pm 1} = \pm\frac{x}{2} + \frac{1}{2a}\ln(\sin ax \pm \cos ax).$$

417 $$\int \frac{\tan ax\, dx}{\tan ax \pm 1} = \frac{x}{2} \mp \frac{1}{2a}\ln(\sin ax \pm \cos ax)$$

21.7.3.5 Integrale mit Kotangensfunktion

418 $$\int \cot ax\, dx = \frac{1}{a}\ln\sin ax$$

419. $$\int \cot^2 ax\, dx = -\frac{\cot ax}{a} - x$$

420 $$\int \cot^3 ax\, dx = -\frac{1}{2a}\cot^2 ax - \frac{1}{a}\ln\sin ax.$$

421 $$\int \cot^n ax\, dx = -\frac{1}{a(n-1)}\cot^{n-1} ax - \int \cot^{n-2} ax\, dx \qquad (n \neq 1).$$

422 $\int x \cot ax\,dx = \frac{x}{a} - \frac{ax^3}{9} - \frac{a^3x^5}{225} - \cdots - \frac{2^{2n}B_n a^{2n-1}x^{2n+1}}{(2n+1)!} - \cdots.$

Mit B_n sind die BERNOULLIschen Zahlen (s 7 2.4.2, S 428) bezeichnet.

423 $\int \frac{\cot ax\,dx}{x} = -\frac{1}{ax} - \frac{ax}{3} - \frac{(ax)^3}{135} - \frac{2(ax)^5}{4725} - \cdots - \frac{2^{2n}B_n(ax)^{2n-1}}{(2n-1)(2n)!} - \cdots.$

424. $\int \frac{\cot^n ax}{\sin^2 ax}dx = -\frac{1}{a(n+1)}\cot^{n+1} ax \qquad (n \neq -1).$

425. $\int \frac{dx}{1 \pm \cot ax} = \int \frac{\tan ax\,dx}{\tan ax \pm 1}$ (s. Nr.417)

21.7.4 Integrale anderer transzendenter Funktionen

21.7.4.1 Integrale mit Hyperbelfunktionen

426. $\int \sinh ax\,dx = \frac{1}{a}\cosh ax.$

427 $\int \cosh ax\,dx = \frac{1}{a}\sinh ax.$

428 $\int \sinh^2 ax\,dx = \frac{1}{2a}\sinh ax \cosh ax - \frac{1}{2}x.$

429. $\int \cosh^2 ax\,dx = \frac{1}{2a}\sinh ax \cosh ax + \frac{1}{2}x$

430 $\int \sinh^n ax\,dx$

$$= \frac{1}{an}\sinh^{n-1} ax \cosh ax - \frac{n-1}{n}\int \sinh^{n-2} ax\,dx \qquad (\text{für } n > 0),$$
$$= \frac{1}{a(n+1)}\sinh^{n+1} ax \cosh ax - \frac{n+2}{n+1}\int \sinh^{n+2} ax\,dx \qquad (\text{für } n < 0)\ (n \neq -1).$$

431. $\int \cosh^n ax\,dx$

$$= \frac{1}{an}\sinh ax \cosh^{n-1} ax + \frac{n-1}{n}\int \cosh^{n-2} ax\,dx \qquad (\text{für } n > 0),$$
$$= -\frac{1}{a(n+1)}\sinh ax \cosh^{n+1} ax + \frac{n+2}{n+1}\int \cosh^{n+2} ax\,dx \qquad (\text{für } n < 0)\ (n \neq -1)$$

432 $\int \frac{dx}{\sinh ax} = \frac{1}{a}\ln\tanh\frac{ax}{2}.$

433. $\int \frac{dx}{\cosh ax} = \frac{2}{a}\arctan e^{ax}.$

434 $\int x \sinh ax\,dx = \frac{1}{a}x\cosh ax - \frac{1}{a^2}\sinh ax$

435 $\int x \cosh ax\,dx = \frac{1}{a}x\sinh ax - \frac{1}{a^2}\cosh ax.$

436. $\int \tanh ax\,dx = \frac{1}{a}\ln\cosh ax$

437 $\int \coth ax\, dx = \frac{1}{a} \ln \sinh ax$

438 $\int \tanh^2 ax\, dx = x - \frac{\tanh ax}{a}$

439. $\int \coth^2 ax\, dx = x - \frac{\coth ax}{a}.$

440 $\int \sinh ax \sinh bx\, dx = \frac{1}{a^2 - b^2}(a \sinh bx \cosh ax - b \cosh bx \sinh ax) \qquad (a^2 \neq b^2).$

441. $\int \cosh ax \cosh bx\, dx = \frac{1}{a^2 - b^2}(a \sinh ax \cosh bx - b \sinh bx \cosh ax) \qquad (a^2 \neq b^2)$

442 $\int \cosh ax \sinh bx\, dx = \frac{1}{a^2 - b^2}(a \sinh bx \sinh ax - b \cosh bx \cosh ax) \qquad (a^2 \neq b^2).$

443 $\int \sinh ax \sin ax\, dx = \frac{1}{2a}(\cosh ax \sin ax - \sinh ax \cos ax)$

444 $\int \cosh ax \cos ax\, dx = \frac{1}{2a}(\sinh ax \cos ax + \cosh ax \sin ax)$

445 $\int \sinh ax \cos ax\, dx = \frac{1}{2a}(\cosh ax \cos ax + \sinh ax \sin ax).$

446 $\int \cosh ax \sin ax\, dx = \frac{1}{2a}(\sinh ax \sin ax - \cosh ax \cos ax)$

21.7.4.2 Integrale mit Exponentialfunktionen

447 $\int e^{ax}\, dx = \frac{1}{a} e^{ax}$

448 $\int x e^{ax}\, dx = \frac{e^{ax}}{a^2}(ax - 1).$

449 $\int x^2 e^{ax}\, dx = e^{ax} \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3} \right).$

450. $\int x^n e^{ax}\, dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax}\, dx.$

451 $\int \frac{e^{ax}}{x}\, dx = \ln x + \frac{ax}{1 \cdot 1!} + \frac{(ax)^2}{2 \cdot 2!} + \frac{(ax)^3}{3 \cdot 3!} + \cdots$

Das bestimmte Integral $\int\limits_{-\infty}^{x} \frac{e^t}{t}\, dt$ nennt man Integralexponentialfunktion (s. 8.2.5,**4.**, S 478) und bezeichnet es mit $\mathrm{Ei}(x)$. Für $x > 0$ divergiert dieses Integral im Punkt $t = 0$, in diesem Falle versteht man unter $\mathrm{Ei}(x)$ den Hauptwert des uneigentlichen Integrals (s. 8.2.5,**4.**,S 478).

$$\int\limits_{-\infty}^{x} \frac{e^t}{t}\, dt = C + \ln|x| + \frac{x}{1 \cdot 1!} + \frac{x^2}{2 \cdot 2!} + \frac{x^3}{3 \cdot 3!} + \cdots + \frac{x^n}{n \cdot n!} + \cdots$$

Mit C ist die EULERsche Konstante (s 8 2.5,**2.**, S. 477) bezeichnet

452. $\int \frac{e^{ax}}{x^n}\, dx = \frac{1}{n-1} \left(-\frac{e^{ax}}{x^{n-1}} + a \int \frac{e^{ax}}{x^{n-1}} dx \right) \qquad (n \neq 1).$

453. $\int \frac{dx}{1+e^{ax}} = \frac{1}{a} \ln \frac{e^{ax}}{1+e^{ax}}.$

454. $\int \frac{dx}{b+ce^{ax}} = \frac{x}{b} - \frac{1}{ab} \ln(b+ce^{ax}).$

455. $\int \frac{e^{ax}\,dx}{b+ce^{ax}} = \frac{1}{ac} \ln(b+ce^{ax}).$

456. $\int \frac{dx}{be^{ax}+ce^{-ax}} = \frac{1}{a\sqrt{bc}} \arctan\left(e^{ax}\sqrt{\frac{b}{c}}\right) \qquad (bc > 0)\,,$

$= \frac{1}{2a\sqrt{-bc}} \ln \frac{c+e^{ax}\sqrt{-bc}}{c-e^{ax}\sqrt{-bc}} \qquad (bc < 0)\,.$

457 $\int \frac{xe^{ax}\,dx}{(1+ax)^2} = \frac{e^{ax}}{a^2(1+ax)}.$

458. $\int e^{ax} \ln x\,dx = \frac{e^{ax}\ln x}{a} - \frac{1}{a}\int \frac{e^{ax}}{x}\,dx\,;$ (s. Nr.451)

459 $\int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx).$

460. $\int e^{ax} \cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx).$

461. $\int e^{ax} \sin^n x\,dx = \frac{e^{ax}\sin^{n-1} x}{a^2+n^2}(a \sin x - n \cos x)$

$+\frac{n(n-1)}{a^2+n^2}\int e^{ax} \sin^{n-2} x\,dx\,;$ (s. Nr.447 und 459).

462. $\int e^{ax} \cos^n x\,dx = \frac{e^{ax}\cos^{n-1} x}{a^2+n^2}(a \cos x + n \sin x)$

$+\frac{n(n-1)}{a^2+n^2}\int e^{ax} \cos^{n-2} x\,dx\,,$ (s. Nr 447 und 460).

463. $\int xe^{ax} \sin bx\,dx = \frac{xe^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) - \frac{e^{ax}}{(a^2+b^2)^2}[(a^2-b^2) \sin bx - 2ab \cos bx]$

464 $\int xe^{ax} \cos bx\,dx = \frac{xe^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) - \frac{e^{ax}}{(a^2+b^2)^2}[(a^2-b^2) \cos bx + 2ab \sin bx]$

21.7.4.3 Integrale mit logarithmischen Funktionen

465 $\int \ln x\,dx = x \ln x - x.$

466. $\int (\ln x)^2\,dx = x(\ln x)^2 - 2x \ln x + 2x.$

467. $\int (\ln x)^3\,dx = x(\ln x)^3 - 3x(\ln x)^2 + 6x \ln x - 6x.$

468 $\int (\ln x)^n\,dx = x(\ln x)^n - n\int (lnx)^{n-1}\,dx \qquad (n \neq -1).$

469. $\int \frac{dx}{\ln x} = \ln\ln x + \ln x + \frac{(\ln x)^2}{2\cdot 2!} + \frac{(\ln x)^3}{3\cdot 3!} + \cdots .$

Das bestimmte Integral $\int\limits_0^x \frac{dt}{\ln t}$ nennt man Integrallogarithmus (s. 8.2.5,**3.**, S. 477) und bezeichnet es mit Li(x) Für $x > 1$ divergiert dieses Integral im Punkt $t = 1$ In diesem Fall versteht man unter Li(x) den Hauptwert des uneigentlichen Integrals (s. 8.2.5,**3.**, S. 477)
Der Integrallogarithmus hängt mit der Integralexponentialfunktion (s. 8.2.5,**4.**, S. 478) zusammen. $\mathrm{Li}(x) = \mathrm{Ei}(\ln x)$

470. $\int \frac{dx}{(\ln x)^n} = -\frac{x}{(n-1)(\ln x)^{n-1}} + \frac{1}{n-1}\int \frac{dx}{(\ln x)^{n-1}}$ $(n \neq 1)$; (s. Nr 469)

471 $\int x^m \ln x\,dx = x^{m+1}\left[\frac{\ln x}{m+1} - \frac{1}{(m+1)^2}\right]$ $(m \neq -1)$.

472. $\int x^m(\ln x)^n\,dx = \frac{x^{m+1}(\ln x)^n}{m+1} - \frac{n}{m+1}\int x^m(\ln x)^{n-1}\,dx$ $(m \neq -1,\ n \neq -1;$ (s. Nr.470).

473 $\int \frac{(\ln x)^n}{x}\,dx = \frac{(\ln x)^{n+1}}{n+1}.$

474. $\int \frac{\ln x}{x^m}\,dx = -\frac{\ln x}{(m-1)x^{m-1}} - \frac{1}{(m-1)^2 x^{m-1}}$ $(m \neq 1)$.

475 $\int \frac{(\ln x)^n}{x^m}\,dx = -\frac{(\ln x)^n}{(m-1)x^{m-1}} + \frac{n}{m-1}\int \frac{(\ln x)^{n-1}}{x^m}dx$ $(m \neq 1)$, (s. Nr.474).

476. $\int \frac{x^m\,dx}{\ln x} = \int \frac{e^{-y}}{y}\,dy$ mit $y = -(m+1)\ln x$, (s. Nr.451)

477. $\int \frac{x^m\,dx}{(\ln x)^n} = -\frac{x^{m+1}}{(n-1)(\ln x)^{n-1}} + \frac{m+1}{n-1}\int \frac{x^m\,dx}{(\ln x)^{n-1}}$ $(n \neq 1)$.

478. $\int \frac{dx}{x\ln x} = \ln\ln x.$

479. $\int \frac{dx}{x^n \ln x} = \ln\ln x - (n-1)\ln x + \frac{(n-1)^2(\ln x)^2}{2\cdot 2!} - \frac{(n-1)^3(\ln x)^3}{3\cdot 3!} + \cdots .$

480. $\int \frac{dx}{x(\ln x)^n} = \frac{-1}{(n-1)(\ln x)^{n-1}}$ $(n \neq 1)$.

481 $\int \frac{dx}{x^p(\ln x)^n} = \frac{-1}{x^{p-1}(n-1)(\ln x)^{n-1}} - \frac{p-1}{n-1}\int \frac{dx}{x^p(\ln x)^{n-1}}$ $(n \neq 1)$.

482. $\int \ln\sin x\,dx = x\ln x - x - \frac{x^3}{18} - \frac{x^5}{900} - \cdots - \frac{2^{2n-1}B_n x^{2n+1}}{n(2n+1)!} - \cdots .$

Mit B_n sind die BERNOULLIschen Zahlen (s. 7 2.4 2, S 428) bezeichnet.

483. $\int \ln\cos x\,dx = -\frac{x^3}{6} - \frac{x^5}{60} - \frac{x^7}{315} - \cdots - \frac{2^{2n-1}(2^{2n}-1)B_n}{n(2n+1)!}x^{2n+1} - \cdots .$

484 $\int \ln\tan x\,dx = x\ln x - x + \frac{x^3}{9} + \frac{7x^5}{450} + \cdots + \frac{2^{2n}(2^{2n-1}-1)B_n}{n(2n+1)!}x^{2n+1} + \cdots$

485 $\int \sin\ln x\,dx = \frac{x}{2}(\sin\ln x - \cos\ln x)$

486. $\int \cos\ln x\,dx = \frac{x}{2}(\sin\ln x + \cos\ln x).$

487. $\int e^{ax}\ln x\,dx = \frac{1}{a}e^{ax}\ln x - \frac{1}{a}\int \frac{e^{ax}}{x}dx\,,$ (s Nr 451)

21.7.4.4 Integrale mit inversen trigonometrischen Funktionen

488. $\int \arcsin\frac{x}{a}\,dx = x\arcsin\frac{x}{a} + \sqrt{a^2-x^2}.$

489 $\int x\arcsin\frac{x}{a}\,dx = \left(\frac{x^2}{2} - \frac{a^2}{4}\right)\arcsin\frac{x}{a} + \frac{x}{4}\sqrt{a^2-x^2}.$

490 $\int x^2\arcsin\frac{x}{a}\,dx = \frac{x^3}{3}\arcsin\frac{x}{a} + \frac{1}{9}(x^2+2a^2)\sqrt{a^2-x^2}$

491 $\int \frac{\arcsin\frac{x}{a}\,dx}{x} = \frac{x}{a} + \frac{1}{2\cdot 3\cdot 3}\frac{x^3}{a^3} + \frac{1\cdot 3}{2\cdot 4\cdot 5\cdot 5}\frac{x^5}{a^5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7\cdot 7}\frac{x^7}{a^7} + \cdots$

492 $\int \frac{\arcsin\frac{x}{a}\,dx}{x^2} = -\frac{1}{x}\arcsin\frac{x}{a} - \frac{1}{a}\ln\frac{a+\sqrt{a^2-x^2}}{x}.$

493. $\int \arccos\frac{x}{a}\,dx = x\arccos\frac{x}{a} - \sqrt{a^2-x^2}$

494 $\int x\arccos\frac{x}{a}\,dx = \left(\frac{x^2}{2} - \frac{a^2}{4}\right)\arccos\frac{x}{a} - \frac{x}{4}\sqrt{a^2-x^2}.$

495. $\int x^2\arccos\frac{x}{a}\,dx = \frac{x^3}{3}\arccos\frac{x}{a} - \frac{1}{9}(x^2+2a^2)\sqrt{a^2-x^2}$

496 $\int \frac{\arccos\frac{x}{a}\,dx}{x} = \frac{\pi}{2}\ln x - \frac{x}{a} - \frac{1}{2\cdot 3\ \ 3}\frac{x^3}{a^3} - \frac{1\cdot 3}{2\cdot 4\cdot 5\cdot 5}\frac{x^5}{a^5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7\cdot 7}\frac{x^7}{a^7} - \cdots$

497. $\int \frac{\arccos\frac{x}{a}\,dx}{x^2} = -\frac{1}{x}\arccos\frac{x}{a} + \frac{1}{a}\ln\frac{a+\sqrt{a^2-x^2}}{x}$

498 $\int \arctan\frac{x}{a}\,dx = x\arctan\frac{x}{a} - \frac{a}{2}\ln(a^2+x^2).$

499. $\int x\arctan\frac{x}{a}\,dx = \frac{1}{2}(x^2+a^2)\arctan\frac{x}{a} - \frac{ax}{2}$

500 $\int x^2\arctan\frac{x}{a}\,dx = \frac{x^3}{3}\arctan\frac{x}{a} - \frac{ax^2}{6} + \frac{a^3}{6}\ln(a^2+x^2)$

501. $\int x^n\arctan\frac{x}{a}\,dx = \frac{x^{n+1}}{n+1}\arctan\frac{x}{a} - \frac{a}{n+1}\int\frac{x^{n+1}\,dx}{a^2+x^2}$ $(n \neq -1)$

502 $\int \frac{\arctan\frac{x}{a}\,dx}{x} = \frac{x}{a} - \frac{x^3}{3^2a^3} + \frac{x^5}{5^2a^5} - \frac{x^7}{7^2a^7} + \cdots$ $(|x| < |a|).$

503. $\int \frac{\arctan \frac{x}{a}\,dx}{x^2} = -\frac{1}{x}\arctan\frac{x}{a} - \frac{1}{2a}\ln\frac{a^2+x^2}{x^2}.$

504 $\int \frac{\arctan \frac{x}{a}\,dx}{x^n} = -\frac{1}{(n-1)x^{n-1}}\arctan\frac{x}{a} + \frac{a}{n-1}\int\frac{dx}{x^{n-1}(a^2+x^2)} \qquad (n \neq 1)$

505. $\int \operatorname{arccot}\frac{x}{a}\,dx = x\operatorname{arccot}\frac{x}{a} + \frac{a}{2}\ln(a^2+x^2)$

506. $\int x\operatorname{arccot}\frac{x}{a}\,dx = \frac{1}{2}(x^2+a^2)\operatorname{arccot}\frac{x}{a} + \frac{ax}{2}$

507 $\int x^2\operatorname{arccot}\frac{x}{a}\,dx = \frac{x^3}{3}\operatorname{arccot}\frac{x}{a} + \frac{ax^2}{6} - \frac{a^3}{6}\ln(a^2+x^2)$

508. $\int x^n\operatorname{arccot}\frac{x}{a}\,dx = \frac{x^{n+1}}{n+1}\operatorname{arccot}\frac{x}{a} + \frac{a}{n+1}\int\frac{x^{n+1}dx}{a^2+x^2} \qquad (n \neq -1).$

509. $\int \frac{\operatorname{arccot}\frac{x}{a}\,dx}{x} = \frac{\pi}{2}\ln x - \frac{x}{a} + \frac{x^3}{3^2a^3} - \frac{x^5}{5^2a^5} - \frac{x^7}{7^2a^7} - \cdots$

510. $\int \frac{\operatorname{arccot}\frac{x}{a}\,dx}{x^2} = -\frac{1}{x}\operatorname{arccot}\frac{x}{a} + \frac{1}{2a}\ln\frac{a^2+x^2}{x^2}.$

511 $\int \frac{\operatorname{arccot}\frac{x}{a}\,dx}{x^n} = -\frac{1}{(n-1)x^{n-1}}\operatorname{arccot}\frac{x}{a} - \frac{a}{n-1}\int\frac{dx}{x^{n-1}(a^2+x^2)} \qquad (n \neq 1).$

21.7.4.5 Integrale mit inversen Hyperbelfunktion

512. $\int \operatorname{Arsinh}\frac{x}{a}\,dx = x\operatorname{Arsinh}\frac{x}{a} - \sqrt{x^2+a^2}.$

513 $\int \operatorname{Arcosh}\frac{x}{a}\,dx = x\operatorname{Arcosh}\frac{x}{a} - \sqrt{x^2-a^2}.$

514 $\int \operatorname{Artanh}\frac{x}{a}\,dx = x\operatorname{Artanh}\frac{x}{a} + \frac{a}{2}\ln(a^2-x^2)$

515 $\int \operatorname{Arcoth}\frac{x}{a}\,dx = x\operatorname{Arcoth}\frac{x}{a} + \frac{a}{2}\ln(x^2-a^2).$