An Effective Stable Domain Model of the Calculus of Construction Extended by Strong Sums and Recursive Definitions

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Abstract

We present a purely domain-theoretic model of Coquand and Huet's Calculus of Construction [3], which is one of the most powerful type systems proposed in the literature.

The well-formed expressions of its language are divided into three levels: *Terms*, *Types*, and *Orders*. Terms are the elements of Types, while the elements of Orders are called *Operators*. There is a special Order constant TYPE denoting the collection of all Types. Orders are closed under both, the dependent product and the dependent sum of Order families indexed over an Order or a Type. In the same way, Types are closed under the dependent product as well as the dependent sum of Type families indexed over an Order or a Type. In addition, we allow recursive definitions at all three levels.

As a subsystem the calculus contains Girard-Reynold's polymorphic λ-calculus [4, 5, 7]. Building upon ideas of Girard [6], Coquand et al. [2] presented a model of this subcalculus in which Types are interpreted as dI-domains.

It is well known that each such domain can be represented as the state set of a stable event structure [1]. In our model, we interpret Types as stable event structures, and in order not to have to deal with too many isomorphic copies we restrict ourselves to stable event structures with only natural numbers as elements. With respect to a natural substructure relation, the set $\mathcal{W}$ of all such event structures turns out to be a locally distributive stable ω-bifinite domain. Here, *locally distributive* means that every principal ideal is distributive.

Stable bifinite domains are algebraic and can thus be obtained (up to isomorphism) from their compact elements by ideal completion. We interpret Orders as partially ordered sets that have only natural numbers as elements and the ideal completion of which is a locally distributive stable ω-bifinite domain. Again, a natural substructure relation can be defined so that the set $\mathcal{B}$ of all such partial orders is a locally distributive ω-algebraic L-domain which satisfies Berry's axiom I.

Expressions of the calculus are constructed with respect to an environment, called *context*, in which the free variables are declared. It turns out that contexts can be interpreted by locally distributive ω-algebraic L-domains that satisfy Berry's axiom I.

For a stable event structure $A \in \mathcal{W}$ let $\mathcal{S}(A)$ be its state space and for a partial order $B \in \mathcal{B}$ let $\mathcal{R}(B)$ be its ideal completion. Then the model is as follows:

- A Type expression α with free variables declared in context Γ is interpreted as a stable map from $[\![\Gamma]\!]$ into $\mathcal{W}$.

- A Term expression t of Type α with free variables declared in context Γ is interpreted as a stable family $[\![t]\!]$ with $[\![t]\!](x) \in \mathcal{S}([\![\alpha]\!](x))$, for $x \in [\![\Gamma]\!]$.
- An Order expression σ with free variables declared in context Γ is interpreted as a stable map from $[\![\Gamma]\!]$ into $\mathcal{B}$.
- An Operator expression T of Order σ with free variables declared in context Γ is interpreted as a stable family $[\![T]\!]$ with $[\![T]\!](x) \in \mathcal{R}([\![\sigma]\!](x))$, for $x \in [\![\Gamma]\!]$.

By this way we can also interpret an extension of the language by disjoint sums of Types and Orders.

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