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On Some Problems in Computable Topology

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Abstract

Computations in spaces like the real numbers are not done on the points of the space itself but on some representation. If one considers only computable points, i.e., points that can be approximated in a computable way, finite objects as the natural numbers can be used for this. In the case of the real numbers such an indexing can e.g. be obtained by taking the Gödel numbers of those total computable functions that enumerate a fast Cauchy sequence of rational numbers. Obviously, the numbering is only a partial map. It will be seen that this is not a consequence of a bad choice, but is so by necessity. The paper will discuss some consequences. All is done in a rather general topological framework.

1 Introduction

The wish and the need to compute with real numbers has been one of the driving forces for the development of large parts of analytical mathematics. Following the *Grundlagenkrise*, in order to develop analysis in a constructive way also approaches based on one or the other of the newly found formalizations of the notion of algorithm were put forward. The problem to define what is a computable real number was indeed Turing's [43, 44] main motivation for the introduction of his machine model.

Today computable analysis is an active research area in theoretical computer science. Among others, the goal is to develop a way of computing with real numbers which does not suffer from such deficiencies as the unpredictable propagation of rounding errors. To achieve this one tries to extend the usual operations on the reals to a class of rational intervals approximating the reals. This larger structure is then also used to interpret the data type `real` used in programming languages. The aim is to use formal verification tools for reasoning about programs that compute with real numbers.

Every decreasing sequence of rational intervals, one properly contained in the other, so that the associated sequence of interval lengths tends to 0, uniquely determines a real number. Rational intervals are easy to code. If there is a total recursive functions that enumerates the codes of such a sequence, the corresponding real number is computable and any Gödel number of the recursive function is called an index of it.

As follows from the definition, the operation of taking the limit of a decreasing sequence of rational intervals is effective with respect to the numbering (or indexing) thus obtained. Moreover, we can enumerate all intervals containing a given computable real, uniformly in

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any of its indices. These are two useful properties. On the other hand, the numbering is only a partial map. Its domain of definition is at least Π_2^0-hard. We will see that this is not a consequence of a clumsy definition: Any indexing of the computable reals having the just mentioned properties must be partial.

A great part of the nowadays theory of numberings has been developed by the Russian school of computability theory (cf. [11, 13, 14, 15, 17]). In these studies only total numberings have been considered so far. This is legitimate as long as one has only numberings of algebraic structures in mind. As we have just seen, the situation is completely different in the case of topological spaces such as the computable reals. Here, the canonical numberings are only partial maps.

The situation with partial numberings is more complicated than the one with only total indexings, in many respects. Typical notions in numbering theory require the existence of certain witness functions. Such a function can behave well also in the case of arguments for which a given condition is not satisfied. The numbering may e.g. be undefined for such a number. Depending on how we deal with cases like this we obtain notions of different strength, which collapse when restricted to total numberings.

As is well-known, given two numberings of a set, one is reducible to the other, if there is a computable function translating, for every element of the numbered set, an index of this element with respect to the first numbering into an index of the same element with respect to the second numbering. In the case of a partial numbering this function can also map a natural number that is not an index of any element onto an index of some element. Thus, in general, knowing that the result of the translation is an index of some element we cannot conclude that the argument of the translating function is an index of the same element, this time with respect to the first numbering. We can require the translation function to do only such translations—in which case we speak of strong reducibility—, but we do not have to. Hence, we obtain two reducibility notions of different strength. As we will see, the degree structures of the partial numberings of a given set with respect to these two reducibility notions are completely different.

By Rice's Theorem all nontrivial properties of the computable real numbers are undecidable. In order to study how difficult they are one considers their index sets. But the index sets have to be taken with respect to a partial numbering. So, if one succeeds, e.g., to determine the level of the index set of some set X of computable real numbers in the arithmetical hierarchy, say Σ_n^0, one cannot conclude that the index set of the complement of X is in Π_n^0. All one knows is that the complement of the index set of X is in Π_n^0. But in addition to indices of elements in the complement of X, this set contains natural numbers with no computational significance: they do not name any computable real number.

As a way out of this dilemma Shapiro [29] suggested instead of index sets to study pairs of index sets, one of X and one of the complement of X. But as we will see, in some cases one is happy to have only a classification of the index set of X.

The space of all computable real numbers with the induced Euclidian topology is just a special case of the general framework considered in this paper. It includes the more general recursive metric spaces studied by Moschovakis [24] as well as constructive versions of Eršov's A- and f-spaces [9, 12, 13, 16] and Scott's directed-complete partial orders [28, 1, 38]. In contrast to metric spaces the latter classes of spaces satisfy only the rather weak T_0 separation axiom.

We consider second-countable T_0 spaces and assume that there has already been a way to define what are their computable elements. Our spaces contain only countably many elements and come with a numbering of these. As we have already seen, in general we can only expect that such numberings are partial. Moreover, we follow M. B. Smyth's approach

[31] and think of the basic open sets as easy to encode observations that can be made about the computational process determining the elements. Therefore, we let the topological basis be indexed in a total way.

By doing better and better observation we want finally be able to determine every element. Thus, we need a relation of definite refinement between the basic open sets which in many cases will be stronger than set inclusion. In most applications it will be recursively enumerable. As it turns out in these cases, the refinement relation is a relation between the codes of the basic open sets and not between the sets itself.

Therefore, we assume that the indexing of the basic open sets is such that there is a transitive relation on the indices so that the property of being a topological basis holds with respect to this relation instead of just set inclusion. The property of being a base of the topology is a $\forall\exists$ statement. We require it to be realized by a computable function on the involved indices. This leads us to the notion of an effective space.

Note that we think of the topological basis with its numbering and the associated refinement relation as being part of the structure under consideration. We will encounter properties which are not invariant under a change of these being givens, though of course the topology remains the same. This seems to be a typical feature of constructive approaches: constructive notions may depend on how objects are represented.

The paper is organized as follows: Section 2 contains basic definitions. Different kinds of reducibilities between partial numberings are presented.

In Section 3 effective spaces are introduced and conditions on the numberings of their points are discussed. They require that the collection of all basic open sets containing a given point can be enumerated, uniformly in any index of that point. Moreover, from an enumeration of a filter base of basic open sets one can compute an index of the point the filter converges to. This leads us to the notion of an acceptable numbering. Numberings with only the first property are called computable. The partial recursive function involved in the second requirement can of course be defined for other arguments as well and still have indices of points as values. Sometimes one has to demand that it cannot do so. In this case the numbering is called strongly acceptable.

In Section 4 standard examples that satisfy the requirements set up in the preceding section are considered. They include constructive A- and f-spaces, constructive domains, recursive metric spaces, and the computable real numbers.

As is shown in Section 5, the class of effective spaces with (strongly) acceptable numberings is closed under the construction of subspaces, Cartesian products, disjunct unions, as well as inverse limits.

In Section 6 the difficulty of some decision problems is studied. First the membership problem for nonopen sets is considered. The index set of such sets is always productive. As a consequence one obtains Rice's Theorem for connected spaces. Then the problem of deciding for a given point whether it is nonfinite is examined. Here, a point is finite if its neighbourhood filter has a finite base. For a large class of spaces including the computable reals this problem is Π^0_2-complete. In case of strongly acceptably indexed spaces, the index set of any set containing a nonfinite point below which there is no finite point is Π^0_2-hard. Note here that any T_0 space comes with a canonical partial order, the specialization order. It follows that every strongly acceptable numbering of a space with such a nonfinite point cannot be total. In particular, we have that the above mentioned numbering of the computable real numbers is necessarily partial.

In Section 7 the behaviour of the two reducibility relations for partial numberings mentioned earlier is investigated. To this end we consider all partial numberings on a fixed set and the induced degree structures. As already said, they are quite different. If the reducibility is employed that straightforwardly extends the one used for total numberings, then for each numbering there are uncountably many reducible to it. Furthermore, the degrees of partial numberings form a distributive lattice, which in the case of an effective topological space contains the degrees of computable numberings as an ideal. Remember that only countably many total numberings can be reduced to a given numbering and the collection of their degrees is an upper semilattice, which in general is not a lattice. Moreover in the total case, the degree of any Friedberg numbering is minimal in the semilattice ordering. Now, if the fixed set is infinite, there is an infinite descending chain of degrees of Friedberg numberings as well as an uncountable antichain of such degrees below every degree.

In case of the strong reducibility relation the situation is similar to that of total numberings: The degrees form only a semilattice. By applying Eršov's completion construction [11] to a partial numbering one obtains a complete total one. This can be used to establish an isomorphism between the upper semilattice of the degrees of partial numberings with respect to strong reducibility and the upper semilattice of all degrees of complete total numberings.

As we have seen so far, in many cases working with partial numberings is less easy that working with total ones and requires special attention. In addition, when conditions like acceptability have to be satisfied we cannot expect that every partial numbering extends to a total numbering of the same set also fulfilling the conditions. On the other hand, Eršov's completion construction allowed to extend partial numberings, the result being not only total but also complete. The construction requires to enlarge the given set by a finite element. Unfortunately, it does not preserve acceptability. So, the question comes up where it is possible to totalize a given acceptable partial numbering by embedding the corresponding space into a larger one, containing more finite elements. A construction of this kind is presented in Section 8. The larger space is an algebraic constructive domain containing the given space as homeomorphic image. If the given space is constructively complete and T_1, it corresponds exactly to the subspace of maximal elements of the new space. Representations of topological spaces by domains have been considered by many authors, mostly in a nonconstructive environment and under different motivations (cf. e.g. [28, 47, 20, 39, 3, 40, 7, 18, 4, 8, 21, 25, 5]).

Concluding remarks will be found in Section 9.

2 Basic definitions

In what follows, let $\langle\ ,\ \rangle : \omega^2 \to \omega$ be a recursive pairing function with corresponding projections π_1 and π_2 such that $\pi_i(\langle a_1, a_2\rangle) = a_i$, and let D be a standard coding of all finite subsets of natural numbers. Moreover, let $P^{(n)}$ ($R^{(n)}$) denote the set of all n-ary partial (total) recursive functions, and let W_i be the domain of the ith partial recursive function φ_i with respect to some Gödel numbering φ. We let $\varphi_i(a)\downarrow$ mean that the computation of $\varphi_i(a)$ stops and $\varphi_i(a)\downarrow \in C$ that it stops with value in C.

Let S be a nonempty set. A *(partial) numbering* ν of S is a partial map $\nu \colon \omega \rightharpoonup S$ (onto) with domain $\mathrm{dom}(\nu)$. The value of ν at $n \in \mathrm{dom}(\nu)$ is denoted by ν_n. Note that instead of numbering we also say *indexing*.

Definition 2.1 A numbering ν of set S is said to be

1. *negative*, if the set $\{\, \langle m, n\rangle \mid m, n \in \mathrm{dom}(\nu) \wedge \nu_m \neq \nu_n \,\}$ is recursively enumerable (r.e.),

2. *precomplete*, if for any function $g \in P^{(1)}$ there is a function $f \in R^{(1)}$ such that $f(n) \in \mathrm{dom}(\nu)$ and $\nu_{f(n)} = \nu_{g(n)}$, for $n \in \mathrm{dom}(g)$ with $g(n) \in \mathrm{dom}(\nu)$,

3. *complete*, if there is some element $e \in S$, called *special element*, such that for any function $g \in P^{(1)}$ there is a function $f \in R^{(1)}$ such that $f(n) \in \mathrm{dom}(\nu)$, for all $n \in \omega$ with either $n \notin \mathrm{dom}(g)$, or $n \in \mathrm{dom}(g)$ and $g(n) \in \mathrm{dom}(\nu)$, and

$$\nu_{f(n)} = \begin{cases} \nu_{g(n)} & \text{if } n \in \mathrm{dom}(g) \text{ with } g(n) \in \mathrm{dom}(\nu), \\ e & \text{if } n \in \omega \setminus \mathrm{dom}(g). \end{cases}$$

As is shown in[11], ν is precomplete if and only if the recursion theorem holds with respect to ν. (Note that in [11] only total numberings are considered, but the proof is valid also in the partial case.)

A subset X of S is *completely enumerable*, if there is an r.e. set W_n such that $\nu_i \in X$ if and only if $i \in W_n$, for all $i \in \mathrm{dom}(\nu)$. Set $M_n = X$, for any such n and X, and let M_n be undefined, otherwise. Then M is a numbering of the class CE of completely enumerable subsets of S. If W_n is recursive, X is called *completely recursive*. X is *enumerable*, if there is an r.e. set $A \subseteq \mathrm{dom}(\nu)$ such that $X = \{\, \nu_i \mid i \in A \,\}$. Thus, X is enumerable if we can enumerate a subset of the index set of X which contains at least one index for every element of X, whereas X is completely enumerable if we can enumerate all indices of elements of X and perhaps some numbers which are not used as indices by the numbering ν.

Definition 2.2 Let ν and κ be numberings of set S.

1. $\nu \leq \kappa$, read ν is *reducible* to κ, if there is some *witness* function $f \in P^{(1)}$ such that $\mathrm{dom}(\nu) \subseteq \mathrm{dom}(f)$, $f(\mathrm{dom}(\nu)) \subseteq \mathrm{dom}(\kappa)$, and $\nu_a = \kappa_{f(a)}$, for all $a \in \mathrm{dom}(\nu)$.

2. $\nu \leq_s \kappa$, read ν is *strongly reducible* to κ, if $\nu \leq \kappa$ via $f \in P^{(1)}$ so that $\mathrm{dom}(\nu) = f^{-1}(\mathrm{dom}(\kappa))$.

3. $\nu \equiv \kappa$, read ν is *equivalent* to κ, if $\nu \leq \kappa$ and $\kappa \leq \nu$. Similarly for *strong equivalence* $\equiv_s$.

As follows from the definition, if ν is reducible to κ via $f \in P^{(1)}$, then we have that $\nu_a = \kappa_{f(a)}$, for all $a \in \mathrm{dom}(\nu)$, whereas ν is strongly reducible to κ via $f \in P^{(1)}$ exactly if $\nu = \kappa \circ f$, where, if read pointwise, this equality means that either both sides are defined and equal, or both sides are undefined. It follows that $\nu \leq_s \kappa$ via $f \in P^{(1)}$ if and only if for every $s \in S$ and all $i \in \omega$,

$$i \in \nu^{-1}(\{s\}) \Leftrightarrow f(i)\!\downarrow \in \kappa^{-1}(\{s\}),$$

and that $\nu \leq \kappa$ via $f \in P^{(1)}$ if and only if for every $s \in S$ and all $i \in \omega$,

$$i \in \nu^{-1}(\{s\}) \Rightarrow f(i)\!\downarrow \in \kappa^{-1}(\{s\}).$$

This shows that strong reducibility extends Ershov's notion of *pm*-reducibility for sets and families of sets [11] to partial numberings. Moreover, we see that in the case that $\nu \leq \kappa$ it is only required that the witness function f behaves correctly when transforming indices i of elements $s \in S$ with respect to ν into indices $f(i)$ of s with respect to κ. We do not demand that if $f(i)$ in an index of some s with respect to κ, then i must be an index of s with respect to ν. Though in some cases we need be able to reason in this way.

Definition 2.3 Let (S, ν) and (S', ν') be numbered sets and $F\colon S \to S'$. Then F is *effective* if there is a function $f \in P^{(1)}$ such that $f(a)\!\downarrow \in \mathrm{dom}(\nu')$ and $F(\nu_a) = \nu'_{f(a)}$, for all $a \in \mathrm{dom}(\nu)$.

Now, let $\mathcal{T} = (T, \tau)$ be a topological T_0 space with countable basis $\mathcal{B}$. We also write $\tau = \langle \mathcal{B} \rangle$ to express that $\mathcal{B}$ is a countable basis of τ. For any subset X of T, $\mathrm{int}_\tau(X)$ and $\mathrm{cl}_\tau(X)$, respectively, are the interior and the closure of X. An open set X is *regular open*, if $X = \mathrm{int}_\tau(\mathrm{cl}_\tau(X))$, and $\mathcal{T}$ is *semi-regular*, if all $X \in \mathcal{B}$ are regular open. (Note that in [6] a topology is called semi-regular, if it has a basis of regular open sets. Here, we think of a topology as being represented by a fixed basis.)

As is well-known, each point y of a T_0 space is uniquely determined by its neighbourhood filter $\mathcal{N}(y)$ and/or a base of it. A point y is called *finite*, if $\mathcal{N}(y)$ has a finite and hence a singleton base. Moreover, on T_0 spaces there is a canonical partial order, the *specialization order*, which we denote by $\leq_\tau$.

Definition 2.4 Let $\mathcal{T} = (T, \tau)$ be a T_0 space, and y, $z \in T$. $y \leq_\tau z$ if $\mathcal{N}(y) \subseteq \mathcal{N}(z)$.

Let B be a numbering of $\mathcal{B}$. By definition each open set is the union of certain basic open sets. In the context of effective topology one is only interested in enumerable unions. We call an open set $O \in \tau$ *Lacombe* set, if there is an r.e. set $A \subseteq \mathrm{dom}(B)$ such that

$$O = \bigcup \{ B_a \mid a \in A \}.$$

Set $L^\tau_n = \bigcup \{ B_a \mid a \in W_n \}$, if $W_n \subseteq \mathrm{dom}(B)$, and let L^τ_n be undefined, otherwise. Then L^τ is a numbering of the Lacombe sets of τ. Obviously, $B \leq L^\tau$.

Now, we can effectively compare second-countable topologies and say what it means that a map is effectively continuous.

Definition 2.5 Let $\tau = \langle \mathcal{B} \rangle$ and $\eta = \langle \mathcal{C} \rangle$ be a topologies on T, and B and C, respectively, be numberings of $\mathcal{B}$ and $\mathcal{C}$.

1. $\eta \subseteq_e \tau$, read τ is *effectively finer* than η, if $C \leq L^\tau$.

2. $\eta =_e \tau$, read η and τ are *effectively equivalent*, if both $\eta \subseteq_e \tau$ and $\tau \subseteq_e \eta$.

Definition 2.6 For $\lambda = 1, 2$, let $\mathcal{T}^\lambda = (T^\lambda, \tau^\lambda)$ be a second-countable topological T_0 space with basis $\mathcal{B}^\lambda$ and associated numbering B^λ. Then a map $F \colon T^1 \to T^2$ is *effectively continuous*, if there is a function $v \in P^{(1)}$ such that for all $n \in \mathrm{dom}(B^1)$, $v(n)\downarrow \in \mathrm{dom}(L^{\tau^1})$ and $F^{-1}(B^2_n) = L^{\tau^1}_{v(n)}$.

In case F is an embedding, i.e., one-to-one, it is called *effectively homeomorphic*, if both F and its partial inverse $F^{-1} \colon F(T) \to T$ are effectively continuous.

Every indexing of a countable set induces a family of natural topologies on this set. Let T be a countable set with numbering x. A topology η on T is a *Mal'cev topology* [22], if it has a basis $\mathcal{C}$ of completely enumerable subsets of T. Any such basis is called a *Mal'cev basis*. $\mathcal{E} = \langle CE \rangle$ is called *Eršov topology*. All Mal'cev bases on T can be indexed in a uniform canonical way. Let $M^\eta_n = M_n$, if $M_n \in \mathcal{C}$, and let it be undefined, otherwise. Then $\mathcal{E}$ is the effectively finest Mal'cev topology on T.

Beside the Eršov topology there are other important Mal'cev topologies. Obviously, CE is a distributive lattice with respect to union and intersection. For $U \in CE$, let U^* denote its pseudocomplement, that is, the greatest completely enumerable subset of $T \setminus U$, if it exists. U is called *regular*, if U^* and U^{**} both exist and $U^{**} = U$. We say that a topology is a *bi-Mal'cev topology*, if it has a basis of regular sets. Any such basis is called a *bi-Mal'cev basis*. Since the class REG of all regular subsets of T is closed under intersection, it also generates a bi-Mal'cev topology on T, which we denote by $\mathcal{R}$.

Let η be a bi-Mal'cev topology on T. Just as in the general case of all Mal'cev bases, also all bi-Mal'cev bases on T can be indexed in a uniform way. Let to this end $R^{\eta}_{\langle m,n\rangle} = M^{\eta}_m$, if $m \in \mathrm{dom}(M^{\eta})$, $n \in \mathrm{dom}(M)$ and ${M^{\eta}_m}^* = M_n$, and let it be undefined, otherwise. Then $\mathcal{R}$ is the effectively finest bi-Mal'cev topology on T.

The reason for the introduction of bi-Mal'cev topologies is that in certain cases one needs to be able to enumerate not only each basic open set, but to a certain extend also its complement. In general, one cannot expect that the whole complement of a basic open set is completely enumerable. So, one has to decide for which part of it this should be the case.

If η is a topology on T, then a subset X of T is called *weakly decidable*, if its interior and its exterior are both completely enumerable. Obviously, every regular set is weakly decidable with respect to the Eršov topology. This leads to another choice of which part of the complements of basic open sets should be completely enumerable. We say that η is *complemented*, if all of its basic open sets are weakly decidable.

The class of all weakly decidable regular open sets of a topology η on T generates a topology η^* which is coarser than η, but the effectively finest complemented semi-regular topology generated by open sets of topology η; it is said to be the complemented semi-regular topology *associated* with η.

Proposition 2.7 ([34]) *$\mathcal{R}$ is the complemented semi-regular topology associated with the Eršov topology on T, that is, $\mathcal{R} = \mathcal{E}^*$.*

3 Effective spaces

In what follows, let $\mathcal{T} = (T, \tau)$ be a countable topological T_0 space with countable basis $\mathcal{B}$.

At first sight the requirement that $\mathcal{T}$ is countable seems quite restrictive. We think of $\mathcal{T}$ as being the subspace of computable elements of some larger space. There are several approaches to topology that come with natural computability notions for points and maps (cf. e.g. [30, 41, 45, 5]). It allows to assign indices to the computable points in a canonical way so that important properties become computable. In general the notion of computable point is rather complex, mainly harder than Σ^0_1. Consequently, the indexings of the computable points thus obtained are only partial maps.

Contrary to this, in most applications the basic open sets have a simple finite description. By coding the descriptions one obtains a total numbering of the topological basis. For us basic open sets are predicates. Each point is uniquely determined by the collection of all predicates it satisfies, thus the T_0 requirement.

Usually, set inclusion between basic open sets is not completely enumerable. But in the applications we have in mind there is a canonical relation between the descriptions of the basic open sets (respectively, their code numbers), which in many cases is stronger than set inclusion. This relation is r.e. We assume that the topological basis $\mathcal{B}$ comes with a numbering B of its elements and such a relation between the codes.

Definition 3.1 Let $\prec_B$ be a transitive binary relation on ω. We say that:

1. $\prec_B$ is a *strong inclusion*, if for all m, $n \in \mathrm{dom}(B)$, from $m \prec_B n$ it follows that $B_m \subseteq B_n$.

2. $\mathcal{B}$ is a *strong basis*, if $\prec_B$ is a strong inclusion and for all $z \in T$ and m, $n \in \mathrm{dom}(B)$ with $z \in B_m \cap B_n$ there is a number $a \in \mathrm{dom}(B)$ such that $z \in B_a$, $a \prec_B m$ and $a \prec_B n$.

For what follows we assume that $\prec_B$ is a strong inclusion with respect to which $\mathcal{B}$ is a strong basis.

Definition 3.2 Let $\mathcal{T} = (T, \tau)$ be a countable topological T_0 space with countable basis $\mathcal{B}$, and let x and B be numberings of T and $\mathcal{B}$, respectively. Then $\mathcal{T}$ is *effective*, if B is total and the property of being a strong basis holds effectively, which means that there exists a function $sb \in P^{(3)}$ such that for $i \in \mathrm{dom}(x)$ and $m, n \in \omega$ with $x_i \in B_m \cap B_n$, $sb(i, m, n)\downarrow$, $x_i \in B_{sb(i,m,n)}$, $sb(i, m, n) \prec_B m$, and $sb(i, m, n) \prec_B n$.

Obviously, the effectivity of $\mathcal{T}$ is invariant under the equivalence of numberings of T. Note that very often the totality of B can easily be achieved, if the space is *recursively separable*, which means that it has a dense enumerable subset, called its *dense base*.

In applications canonical indexings usually have the important property that all basic open sets B_n are completely enumerable, uniformly in n.

Definition 3.3 Let $\mathcal{T} = (T, \tau)$ be a countable topological T_0 space with countable basis $\mathcal{B}$, and let x and B be numberings of T and $\mathcal{B}$, respectively. We say that x is *computable* if there is some r.e. set L such that for all $i \in \mathrm{dom}(x)$ and $n \in \mathrm{dom}(B)$,

$$\langle i, n\rangle \in L \Leftrightarrow x_i \in B_n.$$

As is readily verified, $\mathcal{T}$ is effective if x is computable, B is total and the strong inclusion relation is r.e.

Since we work with strong inclusion instead of set inclusion, we had to adjust the notion of a topological basis. In the same way we have to modify that of a filter base.

Definition 3.4 Let $\mathcal{H}$ be a filter. A nonempty subset $\mathcal{F}$ of $\mathcal{H}$ is called *strong base* of $\mathcal{H}$ if the following two conditions hold:

1. For all m, $n \in \mathrm{dom}(B)$ with B_m, $B_n \in \mathcal{F}$ there is some index $a \in \mathrm{dom}(B)$ such that $B_a \in \mathcal{F}$, $a \prec_B m$, and $a \prec_B n$.
2. For all $m \in \mathrm{dom}(B)$ with $B_m \in \mathcal{H}$ there some index $a \in \mathrm{dom}(B)$ such that $B_a \in \mathcal{F}$ and $a \prec_B m$.

If x is computable, a strong base of basic open sets can effectively be enumerated for each neighbourhood filter. For effective spaces this can always be done in a normed way [34].

Definition 3.5 An enumeration $(B_{f(a)})_{a\in\omega}$ with $f : \omega \to \omega$ such that $\mathrm{range}(f) \subseteq \mathrm{dom}(B)$ is said to be *normed* if f is decreasing with respect to $\prec_B$. If f is recursive, it is also called *recursive* and any Gödel number of f is said to be an *index* of it.

In case $(B_{f(a)})$ enumerates a strong base of the neighbourhood filter of some point, we say it *converges* to that point.

We want not only to be able to generate normed recursive enumerations of basic open sets that converge to a given point, but conversely, we need also to be able to pass effectively from such enumerations to the point they converge to.

Definition 3.6 Let x be a numbering of T. We say that:

1. x *allows effective limit passing* if there is a function $\mathrm{pt} \in P^{(1)}$ such that, if m is an index of a normed recursive enumeration of basic open sets which converges to some point $y \in T$, then $\mathrm{pt}(m)\downarrow \in \mathrm{dom}(x)$ and $x_{\mathrm{pt}(m)} = y$.

2. x is *acceptable* if it allows effective limit passing and is computable.

Note that numbering x is precomplete exactly if there is a total function $\mathrm{pt} \in R^{(1)}$ witnessing that x allows effective limit passing [33].

By definition the function pt is only required to be have correctly on indices of normed converging enumerations of basic open sets. If for some $m \in \omega$ we have that $\mathrm{pt}(m){\downarrow} \in \mathrm{dom}(x)$, we cannot conclude that m is an index of such an enumeration. Sometimes $(B_{\varphi_m(a)})$ should at least enumerate a strong base of $\mathcal{N}(x_{\mathrm{pt}(m)})$.

Definition 3.7 A numbering x of T is *strongly acceptable*, if it is computable and there is a function $pt \in P^{(1)}$ which witnesses that x allows effective limit passing and is such that for $m \in \omega$ it follows from $pt(m){\downarrow} \in \mathrm{dom}(x)$ that the collection of all sets $B_{\varphi_m(a)}$ $(a \in \mathrm{dom}(\varphi_m))$ is a strong base of $\mathcal{N}(x_{pt(m)})$.

As is shown in [32], not every acceptable numbering is also strongly acceptable. But as we will see later, it is always equivalent to such a numbering. They exist under conditions usually satisfied in applications.

If x is computable, each neighbourhood filter $\mathcal{N}(y)$ has a completely enumerable strong base of basic open sets, namely the set of all B_a with $y \in B_a$. In approaches to computable topology like [30, 45] this is the requirement for being a computable point. If, in addition, $\prec_B$ is r.e., one can construct a strongly acceptable numbering of T that is even precomplete and makes $\mathcal{T}$ effective.

Proposition 3.8 *Let $\mathcal{T}$ be such that the neighbourhood filter of each point has an enumerable strong base of basic open sets. Moreover, let $\prec_B$ be r. e. Then T has a precomplete and strongly acceptable numbering $\hat{x}$. If B is total, $\mathcal{T}$ is effective with respect to this numbering.*

Note that the result is a consequence of [32, Proposition 2.11] and [34, Lemma 2.15]. The definition of $\hat{x}$ is straightforward: if m is an index of a normed recursive enumeration of basic open sets that converges to point $y \in T$, set $\hat{x}_m = y$. Otherwise let $\hat{x}$ be undefined. In some results we have to require that up to strong equivalence T is indexed by $\hat{x}$.

Definition 3.9 A numbering x is said to be a *standard numbering* of T, if it is strongly equivalent to numbering $\hat{x}$.

Indexings which are computable and/or allow effective limit passing are related to each other in the following way.

Lemma 3.10 ([34]) *Let $\mathcal{T}$ be effective. Then for any two numberings x' and x'' of T the following hold:*

1. *If x' is computable and x'' allows effective limit passing, then $x' \leq x''$.*

2. *If x' is computable and $x'' \leq x'$, then x'' is computable.*

3. *If x' allows effective limit passing and $x' \leq x''$, then x'' allows effective limit passing.*

Corollary 3.11 *Let $\mathcal{T}$ be effective and x be acceptable. Then for any numbering x' of T the following hold:*

1. *x' is computable if and only if $x' \leq x$.*

2. *x' allows effective limit passing if and only if $x \leq x'$.*

3. *x' is acceptable if and only if $x' \equiv x$.*

We see that the acceptability notion is invariant under equivalence. The same is true for strong acceptability with respect to strong equivalence.

If $\prec_B$ is r.e., it follows with Proposition 3.8 that for each computable numbering x' of T there is an strongly acceptable numbering x'' of T with $x' \leq x''$. If x' is acceptable, we even have that x' and x'' are equivalent.

For neighbourhood filters of points having an enumerable strong base, we can always construct a normed enumeration of a strong base of the same filter. But not every such enumeration needs to converge. This gives rise to the following completeness notion.

Definition 3.12 A T_0 space $\mathcal{T} = (T, \tau, \mathcal{B}, B, \prec_B)$ with a countable strong basis is *constructively complete*, if each normed recursive enumeration of basic open sets converges.

As we shall see in the next section, the constructive completeness of a space may depend on the choice of the topological basis $\mathcal{B}$ (as well as the numbering B and the strong inclusion relation $\prec_B$ belonging to it).

Proposition 3.13 *Let $\mathcal{T}$ be effective and constructively complete such that all basic open sets are nonempty. Let $\prec_B$ be r.e. and x allow effective limit passing. Then $\mathcal{T}$ is recursively separable.*

Proof: Since $\prec_B$ is r.e., there is some function $t \in R^{(1)}$ so that

$$W_{t(m)} = \{ n \in \omega \mid n \prec_B m \}.$$

Note that each such set is nonempty because all basic open sets are not empty and $\mathcal{B}$ is strong. Let $k \in R^{(1)}$ with

$$\varphi_{k(m)}(0) = m,$$
$$\varphi_{k(m)}(i+1) = \varphi_{r(t(\varphi_{k(m)}(i)))}(0).$$

Here, $r \in R^{(1)}$ so that $\varphi_{r(i)}$ enumerates W_i. Again, since all basic open sets are nonempty and $\mathcal{B}$ is strong, $\varphi_{k(m)}$ is total. Thus, $k(m)$ is an index of a normed recursive enumeration of basic open sets. As $\mathcal{T}$ is constructively complete, it converges to $x_{\mathrm{pt}(k(m))}$. By the construction of function k we have that $x_{\mathrm{pt}(k(m))} \in B_m$. This shows that the set of all points $x_{\mathrm{pt}(k(m))}$ ($m \in \omega$) is an enumerable dense base of T.

4 Special cases

In this section we will consider some important standard examples of effective T_0 spaces.

4.1 Constructive domains

Let $Q = (Q, \sqsubseteq)$ be a partial order with least element. A nonempty subset S of Q is *directed*, if for all y_1, $y_2 \in S$ there is some $u \in S$ with y_1, $y_2 \sqsubseteq u$. The *way-below relation* $\ll$ on Q is defined as follows: $y_1 \ll y_2$ if for every directed subset S of Q the least upper bound of which exists in Q, the relation $y_2 \sqsubseteq \bigsqcup S$ implies the existence of an element $u \in S$ with $y_1 \sqsubseteq u$. Note that $\ll$ is transitive. Elements $y \in Q$ with $y \ll y$ are called *compact*.

A subset Z of Q is a *basis* of Q, if for any $y \in Q$ the set $Z_y = \{ z \in Z \mid z \ll y \}$ is directed and $y = \bigsqcup Z_y$. A partial order that has a basis is called *continuous*. If all elements of Z are compact, Q is said to be *algebraic* and Z is called *algebraic basis*.

Now, assume that Q is countable and let x be an indexing of Q. Then Q is *constructively d-complete*, if each of its enumerable directed subsets has a least upper bound in Q. Let Q be constructively d-complete and continuous with basis Z. Moreover, let β be a total numbering of Z. Then $(Q, \sqsubseteq, Z, \beta, x)$ is said to be a *constructive domain*, if the restriction of the way-below relation to Z as well as all sets Z_y, for $y \in Q$, are completely enumerable with respect to the indexing β and $\beta \leq x$.

The numbering x of Q is said to be *admissible*, if the set $\{ \langle i, j \rangle \mid \beta_i \ll x_j \}$ is r.e. and there is a function $d \in R^{(1)}$ such that for all indices $i \in \omega$ for which $\beta(W_i)$ is directed, $x_{d(i)}$ is the least upper bound of $\beta(W_i)$. As shown in [46], such numberings always exist. They can even be chosen as total.

Partial orders come with several natural topologies. In the applications we have in mind, one is mainly interested in the *Scott topology* σ: a subset X of Q is open in σ, if it is upwards closed with respect to the partial order and intersects each enumerable directed subset of Q of which it contains the least upper bound. In the case of a constructive domain this topology is generated by the sets $B_n = \{ y \in Q \mid \beta_n \ll y \}$ with $n \in \omega$. It follows that $\mathcal{Q} = (Q, \sigma)$ is a countable T_0-space with countable basis. Observe that the partial order on Q coincides with the specialization order defined by the Scott toplogy [19]. Moreover, compactness matches with finiteness. Obviously, every admissible numbering is computable. Since Z is dense in $\mathcal{Q}$ we also obtain that $\mathcal{Q}$ is recursively separable.

Define

$$m \prec_B n \Leftrightarrow \beta_n \ll \beta_m.$$

Then $\prec_B$ is a strong inclusion with respect to which the collection of all B_n is a strong basis. Because the restriction of $\ll$ to Z is completely enumerable, $\prec_B$ is r.e. It follows that $\mathcal{Q}$ is effective. Moreover, it is constructively complete and each admissible indexing allows effective limit passing, i.e., it is acceptable. Conversely, every acceptable numbering of Q is admissible.

Note here that since we have to make use of the effectivity characteristics of the basis, these properties can only be verified if we choose the strong inclusion relation as above and do not use simple set inclusion instead.

The next result is a special case of a characterization theorem for effective spaces in [34] and generalizes the well-known Rice/Shapiro Theorem [27].

Theorem 4.1 *For any admissibly indexed constructive domain,* $\sigma =_e \mathcal{E}$.

A partial order Q is *bounded-complete*, if every bounded subset of Q has a least upper bound in Q. Algebraic, bounded-complete constructive domains are called *constructive Scott domains*, if the restriction of the domain order to Z as well as the boundedness of two elements of Z are completely recursive, and there is a function $\mathrm{su} \in R^{(2)}$ such that for any two bounded elements β_m and β_n, $\beta_{\mathrm{su}(m,n)}$ is their least upper bound.

4.2 Constructive A- and f-spaces

A- and f-spaces have been introduced by Eršov [9, 10, 12, 13, 15] as a more topologically oriented approach to domain theory. They are not required to be complete.

Let $\mathcal{Y} = (Y, \rho)$ be a topological T_0 space. For elements y, $z \in Y$ define $y \ll z$ if $z \in \mathrm{int}_\rho(\{ u \in Y \mid y \leq_\rho u \})$. Then y is finite if and only if $y \ll y$. $\mathcal{Y}$ is an *A-space*, if there is a subset Y_0 of Y satisfying the following three properties:

1. Any two elements of Y_0 which are bounded in Y with respect to the specialization order have a least upper bound in Y_0.

2. The collection of sets $\text{int}_\rho(\{ u \in Y \mid y \leq_\rho u \})$, for $y \in Y_0$, is a basis of topology ρ.

3. For any $y \in Y_0$ and $u \in Y$ with $y \ll u$ there is some $z \in Y_0$ such that $y \ll z$ and $z \ll u$.

Any subset Y_0 of Y with these properties is called *basic subspace*.

Let Y be countable and Y_0 have a numbering β. For $m, n \in \text{dom}(\beta)$ set $B_n = \text{int}_\rho(\{u \in Y \mid \beta_n \leq_\rho u \})$ and define

$$m \prec_B n \Leftrightarrow \beta_n \ll \beta_m.$$

Then $\prec_B$ is a strong inclusion with respect to which $\{ B_n \mid n \in \text{dom}(\beta) \}$ is a strong basis. The A-space $\mathcal{Y}$ with basic subspace Y_0 is *constructive*, if the numbering β is total, the restriction of $\ll$ to Y_0 is completely enumerable, and the neighbourhood filter of each point has an enumerable strong base of basic open sets. As follows from Proposition 3.8, Y has a precomplete standard numbering x such that $\mathcal{Y}$ is effective. Moreover, it is recursively separable with dense basis Y_0.

Theorem 4.2 ([34]) *For every constructive A-space (Y, ρ), $\rho =_e \mathcal{E}$.*

Since the topology ρ of a constructive A-space is not required to be the Scott topology (with respect to $\leq_\rho$), constructive d-completeness is too weak a completeness notion in this case.

Definition 4.3 A constructive A-space $\mathcal{Y}$ is *effectively complete*, if every enumerable directed subset S of Y with the property that for every $z \in S$ there is some $z' \in S$ with $z \ll z'$, has an upper bound $y \in Y$ which is also a limit point of S.

Obviously, given such a set S we can enumerate a subset S' such that any two elements of S' are comparable with respect to $\ll$ and for every $z \in S$ there is some $z' \in S'$ with $z \ll z'$. This gives us the following result.

Proposition 4.4 *A constructive A-space $\mathcal{Y}$ is constructively complete if and only if it is effectively complete.*

Let $\mathcal{Y} = (Y, \rho)$ be again an arbitrary topological T_0-space. An open set V is an *f-set*, if there is an element $z_V \in V$ such that $V = \{ y \in Y \mid z_V \leq_\rho y \}$. The uniquely determined element z_V is called an *f-element.* $\mathcal{Y}$ is an *f-space*, if the following two conditions hold:

1. If U and V are f-sets with nonempty intersection, then $U \cap V$ is also an f-set.

2. The collection of all f-sets is a basis of topology ρ.

An f-space is *constructive*, if the set of all f-elements has a total numbering α such that the restriction of the specialization order to this set as well as the boundedness of two f-elements are completely recursive and there is a function $\text{su} \in R^{(2)}$ such that in the case that α_n and α_m are bounded, $\alpha_{\text{su}(n,m)}$ is their least upper bound, and if the neighbourhood filter of each point has an enumerable base of f-sets.

Every f-space is an A-space with basic subspace the set of all f-elements, which are exactly the finite elements of the space. Moreover, for y, $z \in Y$ with y or z being an f-element, $y \ll z$ if and only if $y \leq_\rho z$. It follows that also every constructive f-space is a constructive A-space.

4.3 Constructive metric spaces

Let $\mathbb{R}$ denote the set of all real numbers, and let ν be some canonical total indexing of the rational numbers. Then a real number z is said to be *computable*, if there is a function $f \in R^{(1)}$ such that for all $m, n \in \omega$ with $m \leq n$, the inequality $|\nu_{f(m)} - \nu_{f(n)}| < 2^{-m}$ holds and $z = \lim_m \nu_{f(m)}$. Any Gödel number of the function f is called an *index* of z. This defines a partial indexing γ of the set $\mathbb{R}_c$ of all computable real numbers.

Now, let $\mathcal{M} = (M, \delta)$ be a separable metric space with range$(\delta) \subseteq \mathbb{R}_c$, and let β be a total numbering of the dense subset M_0. A sequence $(y_a)_{a\in\omega}$ of elements of M_0 is said to be *fast*, if $\delta(y_m, y_n) < 2^{-m}$, for all $m, n \in \omega$ with $m \leq n$. Moreover, (y_a) is *recursive*, if there is some function $f \in R^{(1)}$ such that $y_a = \beta_{f(a)}$, for all $a \in \omega$. Any Gödel number of f is called an *index* of (y_a).

$\mathcal{M}$ is said to be *constructive*, if the restriction of the distance function δ to M_0 is effective, i.e., if there is some function $d \in R^{(2)}$ such that for all $i, j \in \omega$, $\delta(\beta_i, \beta_j) = \gamma_{d(i,j)}$, and each element y of M is the limit of a fast recursive sequence of elements of M_0. If m is the index of such a sequence, set $x_m = y$. Otherwise, let x be undefined. Then x is a numbering of M with respect to which and the indexing γ of the computable real numbers the distance function is effective [32].

As is well-known, the collection of sets $B_{\langle i,m\rangle} = \{ y \in M \mid \delta(\beta_i, y) < 2^{-m} \}$ $(i, m \in \omega)$ is a basis of the canonical Hausdorff topology Δ on M. Because the usual less-than relation on the computable real numbers is completely enumerable [23], it follows that x is computable. A point $y \in M$ is finite if and only if it is isolated [32].

Define

$$\langle i, m\rangle \prec_B \langle j, n\rangle \Leftrightarrow \delta(\beta_i, \beta_j) + 2^{-m} < 2^{-n}.$$

Using the triangle inequality it is readily verified that $\prec_B$ is a strong inclusion and the collection of all B_a is a strong basis. Moreover, $\prec_B$ is r.e. It follows that $\mathcal{M}$ is effective. The numbering x is precomplete and standard [32].

Theorem 4.5 ([32, 34]) *Let $\mathcal{M}$ be a constructive metric space. Then the following statements hold:*

1. *$\mathcal{M}$ is constructively complete if and only if every fast recursive sequence of elements of the dense subset converges.*

2. $\Delta =_e \mathcal{R}$.

Well-known examples of constructive metric spaces include $\mathbb{R}_c^n$ with the Euclidean or the maximum norm, Baire space, that is, the set $R^{(1)}$ of all total recursive functions with the Baire metric [27], and the set ω with the discrete metric. By using an effective version of Weierstraß's approximation theorem [26] and Sturm's theorem [42] it can be shown that $C_c[0, 1]$, the space of all computable functions from $[0, 1]$ to $\mathbb{R}$ [26] with the supremum norm, is a constructive metric space. A proof of this result and further examples can be found in Blanck [4].

As follows from Theorem 4.5 (2), the metric topology on Baire space is also generated by the completely enumerable subsets with completely enumerable complement.

4.4 The computable real numbers

When starting our presentation of the theory of effective topological spaces we took for granted that the spaces under consideration come with a fixed countable strong basis $\mathcal{B}$ of the topology, a total numbering B of $\mathcal{B}$ and a strong inclusion relation. An assumption of this kind seems to be unavoidable in a constructive approach. As a consequence, certain notions may not be invariant under basis change.

Let us consider the space $\mathbb{R}_c$ of all computable real numbers with the induced Euclidean topology. Then $\mathbb{R}_c$ is a constructive metric space, which is constructively complete with respect to the basis introduced above. When dealing with the real numbers we usually use other bases, e.g. the collection $\mathcal{I}$ of all open intervals with rational endpoints. As we shall show now, $\mathbb{R}_c$ is *not* constructively complete with respect to this basis.

Let ν again be a canonical indexing of the rational numbers and $f \in R^{(1)}$ enumerate the set $\{ \langle m, n\rangle \mid \nu_m < \nu_n \}$. Set $I_a = \big(\nu_{\pi_1(f(a))}, \nu_{\pi_2(f(a))}\big)$ and let

$$a \prec_I b \Leftrightarrow \nu_{\pi_1(f(b))} < \nu_{\pi_1(f(a))} \wedge \nu_{\pi_2(f(a))} < \nu_{\pi_2(f(b))}.$$

Then $\prec_B$ is an r.e. strong inclusion with respect to which $\mathcal{I}$ is a strong basis. Moreover, for every $y \in \mathbb{R}_c$, $\mathcal{N}(y)$ has an enumerable strong base of basic open sets. It follows that also in this case $\mathbb{R}_c$ has a precomplete standard numbering so that it is an effective space. In addition, $\mathbb{R}_c$ is recursively separable.

Now, let $m \in \omega$ with $I_{\varphi_m(n)} = \big(-\frac{1}{n}, 1 + \frac{1}{n}\big)$. Then m is an index of a normed recursive enumeration of basic open sets. But this enumeration does not converge.

5 New spaces from old

In this section we study how properties like the effectivity of spaces or the acceptability of numberings are inherited to spaces constructed from given ones.

5.1 Base reduction

By definition a topological basis is allowed to contain the empty set, but sometimes it is useful to exclude this. We will show that under the assumption that $\{ n \in \omega \mid B_n \neq \emptyset \}$ is r.e., we can define a numbering B' of the basis $\mathcal{B}' = \mathcal{B} \setminus \{\emptyset\}$ and a strong inclusion relation $\prec_{B'}$ so that $\mathcal{B}'$ is strong again and $\mathcal{T}$ is effective with respect to the new basis, if it was so with respect to the old one.

Let $f \in R^{(1)}$ enumerate the set of all $n \in \omega$ for which B_n is not empty. Then f has a right inverse $f' \in P^{(1)}$ defined by $f'(a) = \mu n : f(n) = a$. Set $B'_n = B_{f(n)}$ and let

$$m \prec_{B'} n \Leftrightarrow f(m) \prec_B f(n).$$

Lemma 5.1 *Let $\mathcal{T} = (T, \tau)$ be a countable T_0 space with countable basis $\mathcal{B}$, and let x and B be numberings of T and $\mathcal{B}$, respectively. Moreover, let $\prec_B$ be a strong inclusion relation. Then $\prec_{B'}$ is a strong inclusion as well and the following statements hold:*

1. *The identity on T induces an effective homeomorphism between $\mathcal{T} = (T, \tau, \mathcal{B}, B, \prec_B, x)$ and $\mathcal{T}' = (T, \tau, \mathcal{B}', B', \prec_{B'}, x)$.*
2. *If $\mathcal{B}$ is strong, the same is true for $\mathcal{B}'$.*
3. *If $\mathcal{T}$ is effective, then $\mathcal{T}'$ is effective as well.*
4. *If $\mathcal{T}$ is constructively complete, then also $\mathcal{T}'$ is constructively complete.*

Proof: For the proof of the second statement let $z \in T$ with $z \in B'_m \cap B'_n$. Then $z \in B_{f(m)} \cap B_{f(n)}$. Hence, there is some $a \prec_B f(m), f(n)$ with $z \in B_a$. It follows that B_a is not empty. Thus $f'(a)$ is defined, $z \in B'_{f'(a)}$ and $f'(a) \prec_{B'} m, n$. The proof of the other statements is obvious.

Note that the above assumption about $\{ n \in \omega \mid B_n \neq \emptyset \}$ is always fulfilled, if $\mathcal{T}$ is recursively separable and x is computable.

When we introduced the computability and similar notions for the numbering x of T, we did that with respect to a fixed topological basis $\mathcal{B}$ with numbering B and strong inclusion $\prec_B$. All these properties remain unchanged, if we use $\mathcal{B}'$, B' and $\prec_{B'}$ instead.

Lemma 5.2 *Let $\mathcal{T} = (T, \tau)$ be a countable T_0 space with countable basis $\mathcal{B}$, and let x and B be numberings of T and $\mathcal{B}$, respectively. Moreover, let $\prec_B$ be a strong inclusion relation. Then the following statements hold:*

1. *If x is computable with respect to B, it is computable with respect to B' as well.*
2. *If x allows effective limit passing with respect to B and $\prec_B$, it does so also with respect to B' and $\prec_{B'}$.*
3. *If x is (strongly) acceptable with respect to B and $\prec_B$, it is also (strongly) acceptable with respect to B' and $\prec_{B'}$.*
4. *If x is a standard numbering with respect to B and $\prec_B$, it is also a standard numbering with respect to B' and $\prec_{B'}$*

Proof: The proof of the first statement is obvious. For the proof of the second one let $k \in R^{(1)}$ with $\varphi_{k(m)} = f \circ \varphi_m$. Then, if m is an index of a normed recursive enumeration of basic open sets with respect to B' and $\prec_{B'}$, $k(m)$ is an index of the same recursive enumeration which is now normed with respect to B and $\prec_B$. It follows that $\mathrm{pt}' = \mathrm{pt} \circ k$ witnesses that x allows effective limit passing with respect to B' and $\prec_{B'}$.

The third statement is now an easy consequence. For the proof of the last statement define numberings $\bar{x}$ and $\hat{x}$ as follows. If $(B_{\varphi_m(a)})_{a \in \omega}$ and $(B'_{\varphi_m(a)})_{a \in \omega}$ are normed enumerations of a strong base of $\mathcal{N}(y)$, for $y \in T$, set $\bar{x}_m = \hat{x} = y$. Otherwise let both numberings be undefined. Then $\hat{x} \leq_s \bar{x}$ via the just defined function $k \in R^{(1)}$.

Conversely, if $(B_{\varphi_m(a)})$ is a normed enumeration of a strong base of $\mathcal{N}(\bar{x}_m)$, then $B_{\varphi_m(a)}$ is not empty, for all $a \in \omega$. Hence $f'(\varphi_m(a))$ is always defined. Let $k' \in R^{(1)}$ with $\varphi_{k'(m)} = f' \circ \varphi_m$. Then k' witnesses that $\hat{x} \leq_s \bar{x}$. Thus, $\bar{x} \equiv_s \hat{x}$.

Now, if x is a standard numbering with respect to B and $\prec_B$, then $x \equiv_s \bar{x}$. It follows that $x \equiv_s \hat{x}$, which shows that x is also standard with respect to B' and $\prec_{B'}$.

5.2 Subspaces

Let S be a subset of T and τ^S be the induced topology on S. Define $\mathcal{B}^S$ to be the collection of all sets $X \cap S$ with $X \in \mathcal{B}$, set $B^S_n = B_n \cap S$ and let

$$m \prec_{B^S} n \Leftrightarrow m \prec_B n.$$

Finally, let $x^S_i = x_i$, if $i \in \mathrm{dom}(x)$ and $x_i \in S$. In any other case let x^S be undefined. Set $\mathcal{T}^S = (S, \tau^S, \mathcal{B}^S, B^S, \prec_{B^S}, x^S)$.

Lemma 5.3 *Let $\mathcal{T} = (T, \tau)$ be a countable T_0 space with countable basis $\mathcal{B}$, numberings x and B of T and $\mathcal{B}$, respectively, and strong inclusion $\prec_B$. Moreover, let $S \subseteq T$. Then $\prec_{B^S}$ is also a strong inclusion and the following statements hold:*

1. *The canonical embedding of S in T is both effective and effectively continuous.*
2. *If $\mathcal{B}$ is strong, $\mathcal{B}^S$ is also strong.*
3. *If $\mathcal{T}$ is effective, $\mathcal{T}^S$ is effective as well.*
4. *If x is computable, allows effective limit passing, is (strongly) acceptable, or a standard numbering, the same holds for x^S.*

Note here that for numberings $\bar{x}$ and $\hat{x}$ of T with $\bar{x} \leq_s \hat{x}$, we have that $\bar{x}^S \leq_s \hat{x}^S$.

5.3 Disjoint unions

For $\lambda = 1, 2$, let $\mathcal{T}^\lambda = (T^\lambda, \tau^\lambda)$ be a countable T_0 space with countable basis $\mathcal{B}^\lambda$, numberings x^λ and B^λ of T^λ and $\mathcal{B}^\lambda$, respectively, and strong inclusion relation $\prec_{B^\lambda}$. Set

$$T^1 \oplus T^2 = \{1\} \times T^1 \cup \{2\} \times T^2,$$

$$B^{\oplus}_{\langle n,m\rangle} = \begin{cases} \{1\} \times B^1_m & \text{if } n \text{ is odd,} \\ \{2\} \times B^2_m & \text{otherwise} \end{cases} \qquad (n, m \in \omega),$$

$$\langle n, m\rangle \prec_{B^\oplus} \langle n', m'\rangle \Leftrightarrow [n, n' \text{ odd } \wedge m \prec_{B^1} m'] \vee [n, n' \text{ even } \wedge m \prec_{B^2} m'].$$

Let $\mathcal{B}^\oplus$ be the collection of all sets $B^\oplus_a$, for $a \in \omega$. Then $\mathcal{B}^\oplus$ is a basis of the canonical topology $\tau^\oplus$ on $T^1 \oplus T^2$. Moreover, $\prec_{B^\oplus}$ is a strong inclusion relation. Obviously, $\mathcal{T}^1 \oplus \mathcal{T}^2 = (T^1 \oplus T^2, \tau^\oplus)$ is a countable T_0 space again. Set

$$x^{\oplus}_{\langle n,i\rangle} = \begin{cases} (1, x^1_i) & \text{if } n \text{ is odd and } i \in \mathrm{dom}(x^1), \\ (2, x^2_i) & \text{if } n \text{ is even and } i \in \mathrm{dom}(x^2). \end{cases}$$

In any other case let $x^\oplus$ be undefined.

Lemma 5.4 *For $\lambda = 1, 2$, let $\mathcal{T}^\lambda = (T^\lambda, \tau^\lambda)$ be a countable T_0 space with countable basis $\mathcal{B}^\lambda$, numberings x^λ and B^λ of T^λ and $\mathcal{B}^\lambda$, respectively, and strong inclusion relation $\prec_{B^\lambda}$. Then the following statements hold:*

1. *For $\lambda = 1, 2$, the canonical inclusion of T^λ into $T^1 \oplus T^2$ is both effective and effectively continuous.*
2. *If $\mathcal{B}^1$ and $\mathcal{B}^2$ are strong, then $\mathcal{B}^\oplus$ is strong too.*
3. *If $\mathcal{T}^1$ and $\mathcal{T}^2$ are constructively complete, then also $\mathcal{T}^1 \oplus \mathcal{T}^2$ is constructively complete.*
4. *If $\mathcal{T}^1$ and $\mathcal{T}^2$ are effective, then $\mathcal{T}^1 \oplus \mathcal{T}^2$ is effective as well.*
5. *If x^1 and x^2 are both computable, allow effective limit passing, are (strongly) acceptable, or standard numberings, the same holds for $x^\oplus$.*

The above results can be extended to countable unions in a straightforward way. In the case of the last two statements one has to require that the corresponding effectivity conditions hold uniformly, which e.g. means that the witnessing functions in the definition of a standard numbering computably depend on the space index.

5.4 Cartesian products

Let $\mathcal{T}^\lambda$ be as above, for $\lambda = 1, 2$. Set $B^\times_{\langle m,n\rangle} = B^1_m \times B^2_n$, for $m, n \in \omega$, and define

$$\langle m, n\rangle \prec_{B^\times} \langle m', n'\rangle \Leftrightarrow m \prec_{B^1} m' \wedge n \prec_{B^2} n'.$$

Then $\prec_{B^\times}$ is a strong inclusion. Moreover $\mathcal{B}^\times$, the collection of all sets $B^\times_a$ with $a \in \omega$, is basis of the canonical topology $\tau^\times$ on $T^1 \times T^2$. $\mathcal{T}^1 \times \mathcal{T}^2 = (T^1 \times T^2, \tau^\times)$ is again a countable T_0 space. For $(i, j) \in \mathrm{dom}(x^1) \times \mathrm{dom}(x^2)$, set $x^\times_{\langle i,j\rangle} = (x^1_i, x^2_j)$. Otherwise, let $x^\times$ be undefined.

Lemma 5.5 *For $\lambda = 1, 2$, let $\mathcal{T}^\lambda = (T^\lambda, \tau^\lambda)$ be a countable T_0 space with countable basis $\mathcal{B}^\lambda$, numberings x^λ and B^λ of T^λ and $\mathcal{B}^\lambda$, respectively, and strong inclusion relation $\prec_{B^\lambda}$. Then the following statements hold:*

1. *For $\lambda = 1, 2$, the canonical projection of $T^1 \times T^2$ onto T^λ is both effective and effectively continuous.*
2. *If $\mathcal{B}^1$ and $\mathcal{B}^2$ are strong, then $\mathcal{B}^\times$ is strong as well.*
3. *If $\mathcal{T}^1$ and $\mathcal{T}^2$ are constructively complete, then also $\mathcal{T}^1 \times \mathcal{T}^2$ is constructively complete.*
4. *If $\mathcal{T}^1$ and $\mathcal{T}^2$ are effective, then $\mathcal{T}^1 \times \mathcal{T}^2$ is effective too.*
5. *If x^1 and x^2 are both computable, allow effective limit passing, are (strongly) acceptable, or standard numberings, the same holds for $x^\times$.*

5.5 Countable products

Let $f \in R^{(1)}$ enumerate the set $\{ a \in \omega \mid (\forall c, c' \in D_a)[\pi_1(c) = \pi_1(c') \Rightarrow \pi_2(c) = \pi_2(c')] \}$ and for $\lambda \in \omega$, let $\mathcal{T}^\lambda = (T^\lambda, \tau^\lambda)$ be a countable T_0 space with countable basis $\mathcal{B}^\lambda$, numberings x^λ and B^λ of T^λ and $\mathcal{B}^\lambda$, respectively, and strong inclusion relation $\prec_{B^\lambda}$. Set

$$T^\Pi = \{ \bar{z} \in \textstyle\prod_{\lambda\in\omega} T^\lambda \mid (\exists t \in R^{(1)})(\forall \lambda \in \omega)\bar{z}_\lambda = x^\lambda_{t(\lambda)} \}$$

and let $P_\lambda \colon T^\Pi \to T^\lambda$ be the projection on the λ-th component. Moreover, for $a, b \in \omega$, let $B^\Pi_a = \bigcap \{ P^{-1}_\lambda(B^\lambda_n) \mid \langle \lambda, n\rangle \in D_{f(a)} \}$ and define

$$\begin{aligned} a \prec_{B^\Pi} b \Leftrightarrow \pi_1(D_{f(b)}) \subsetneq \pi_1(D_{f(a)}) \wedge (\forall \langle \lambda, n\rangle \in D_{f(b)}) \\ (\exists \langle \lambda', n'\rangle \in D_{f(a)})(\lambda = \lambda' \wedge n' \prec_{B^\lambda} n). \end{aligned}$$

Then $\mathcal{B}^\Pi$, the collection of all sets B^Π_a, for $a \in \omega$, is a basis of the canonical topology τ^Π on T^Π. Furthermore, $\prec_{B^\Pi}$ is a strong inclusion. $\mathcal{T}^\Pi = (T^\Pi, \tau^\Pi)$ is again a countable T_0 space. If $i \in \omega$ such that $\varphi_i(\lambda){\downarrow} \in \mathrm{dom}(x^\lambda)$, for all $\lambda \in \omega$, define x^Π_i by $x^\Pi_i(\lambda) = x^\lambda_{\varphi_i(\lambda)}$. In any other case let x^Π be undefined.

Lemma 5.6 *For $\lambda \in \omega$, let $\mathcal{T}^\lambda = (T^\lambda, \tau^\lambda)$ be a countable T_0 space with countable basis $\mathcal{B}^\lambda$, numberings x^λ and B^λ of T^λ and $\mathcal{B}^\lambda$, respectively, and strong inclusion relation $\prec_{B^\lambda}$. Then the following statements hold:*

1. *For $\lambda \in \omega$, the canonical projection of T^Π onto T^λ is both effective and effectively continuous, uniformly in λ.*

2. *If $\mathcal{B}^\lambda$ is strong, for all $\lambda \in \omega$, then $\mathcal{B}^\Pi$ is also strong.*

3. *If x^λ allows effective limit passing, uniformly in λ, and $\mathcal{T}^\lambda$ is constructively complete, for every $\lambda \in \omega$, then $\mathcal{T}^\Pi$ is constructively complete too.*

4. *If each $\mathcal{T}^\lambda$ is effective, uniformly in λ, then $\mathcal{T}^\Pi$ is effective as well.*

5. *If all x^λ are computable, allow effective limit passing, are (strongly) acceptable, or standard numberings, always uniformly in λ, the same holds for x^Π.*

Proof: The statements follow in a more or less straightforward way. We only show (3). Note that given an index m of a normed recursive enumeration of basic open sets B_a^Π and an index λ, we can effectively find some $a \in \omega$ so that $\lambda \in \pi_1(D_{f(\varphi_m(a))})$. By the definition of $\prec_{B^\Pi}$, we have that $\lambda \in \pi_1(D_{f(\varphi_m(a'))})$, for all $a' \geq a$. Let $k(\lambda, c) = \mu n : \langle \lambda, n \rangle \in D_{f(c)}$ and $n \in \omega$ be such that $\varphi_n(c) = k(\lambda, \varphi_m(c+a))$. Then n is an index of a normed recursive enumeration of basic open sets in space $\mathcal{T}^\lambda$. By assumption there is a function $p \in R^{(1)}$ so that $\varphi_{p(\lambda)}$ witnesses that x^λ allows effective limit passing. Thus, the recursive enumeration with index n converges to $x^\lambda_{\varphi_{p(\lambda)}(n)}$. Let $h \in R^{(1)}$ with $\varphi_{h(m)}(\lambda) = \varphi_{p(\lambda)}(n)$. Then the enumeration with index m converges to $x^\Pi_{h(m)}$.

5.6 Inverse limits

Let **ETOP** (**ETOP$^\mathbf{e}$**) be the category of effective T_0 spaces with (effective and) effectively continuous maps. **ETOP$_\mathbf{co}$**, **ETOP$_\mathbf{lp}$**, **ETOP$_\mathbf{ac}$**, **ETOP$_\mathbf{sa}$**, and **ETOP$_\mathbf{st}$**, respectively, be the full subcategories of spaces with indexings that are computable, allow effective limit passing, are acceptable, strongly acceptable, or standard. Analogous subcategories are defined for **ETOP$^\mathbf{e}$**. Denote the collection of these categories by $\mathfrak{C}$.

An ω-cochain $\mathbb{T} = \mathcal{T}^0 \overset{F_0}{\leftarrow} \mathcal{T}^1 \overset{F_1}{\leftarrow} \mathcal{T}^2 \overset{F_2}{\leftarrow} \ldots$ in **ETOP** is called *effective*, if the effectivity of the $\mathcal{T}^\lambda$ and the effective continuity of the maps F_λ hold uniformly in λ; analogously for the other categories.

Let $\mathbf{C} \in \mathfrak{C}$ and $\mathbb{T}$ be an ω-cochain in $\mathbf{C}$. A cone $(\widetilde{T}, (\widetilde{P}_\lambda)_{\lambda\in\omega})$ to $\mathbb{T}$ is *effective*, if the effective continuity of the maps $\widetilde{P}_\lambda$ holds uniformly in λ. If $\mathbf{C}$ is the category **ETOP$^\mathbf{e}$** or one of its full subcategories, also the effectivity of the $\widetilde{P}_\lambda$ has to hold uniformly. Then an *effective limit* to $\mathbb{T}$ is a terminal object in the category of effective cones to $\mathbb{T}$ in $\mathbf{C}$.

Now, let $\mathbb{T} = (\mathcal{T}^\lambda, F_\lambda)_{\lambda\in\omega}$ be an effective cochain in $\mathbf{C}$ and define $\mathcal{T}^\infty$ by

$$T^\infty = \{\, \bar{z} \in T^\Pi \mid (\forall \lambda \in \omega) \bar{z}_\lambda = F_\lambda(\bar{z}_{\lambda+1}) \,\}.$$

The corresponding topology is induced by the product space topology and its basis with numbering and associated strong inclusion, as well as the indexing of space T^∞ are given in the same way as in Section 5.2.

Let P_λ denote again the canonical projection of T^∞ onto T^λ. Then it follows from Lemmata 5.3 and 5.6 that $(\mathcal{T}^\infty, (P_\lambda)_{\lambda\in\omega})$ is an effective cone to $\mathbb{T}$ in $\mathbf{C}$.

Proposition 5.7 *Let $\mathbf{C} \in \mathfrak{C}$ and $\mathbb{T} = (\mathcal{T}^\lambda, F_\lambda)_{\lambda\in\omega}$ be a cochain in $\mathbf{C}$. Then $(\mathcal{T}^\infty, (P_\lambda)_{\lambda\in\omega})$ is an effective limit to $\mathbb{T}$ in $\mathbf{C}$.*

6 Some decision problems

If ν is a total numbering of some set S, then for each subset X of S there is a predicate on the natural numbers which is true for natural number n, if $\nu_n \in X$, and false, otherwise. In case that ν is a partial numbering there is only a partial predicate, being true for $n \in \omega$, if $\nu_n \in X$, false, if $\nu_n \notin X$, and undefined, otherwise. We represent such partial predicates by pairs of disjoint sets of natural numbers. Let $\Omega(X) = \{ n \in \mathrm{dom}(\nu) \mid \nu_n \in X \}$, then $(\Omega(X), \Omega(\overline{X}))$ where $\overline{X}$ is the complement of X is the partial predicate corresponding to X. If ν is a total numbering, we identify the predicate $(\Omega(X), \Omega(\overline{X}))$ with the set $\Omega(X)$.

Definition 6.1 Let (A_1, A_2) and (B_1, B_2) be partial predicates. $(A_1, A_2) \leq_m (B_1, B_2)$, read (A_1, A_2) is *many-one reducible* to (B_1, B_2), if there is a function $f \in P^{(1)}$ such that for $i = 1$, 2 and all $a \in A_1 \cup A_2$, if $a \in A_i$ then $f(a)\downarrow \in B_i$.

This definition is due to Shapiro [29]. In the case that A_2 and B_2 respectively are the complements of A_1 and B_1, it reduces to the well-known many-one reducibility. We denote both kinds of reducibility by $\leq_m$. Moreover, $\leq_1$ denotes one-one reducibility, $\equiv_m$ many-one equivalence, and $\equiv_1$ one-one equivalence.

Definition 6.2 Let $\mathcal{S}$ be a class of sets of natural numbers and (A_1, A_2) a partial predicate.

1. (A_1, A_2) is *$\mathcal{S}$-hard*, if for every $B \in \mathcal{S}$ we have that $(B, \overline{B}) \leq_m (A_1, A_2)$.
2. (A_1, A_2) is *potentially in $\mathcal{S}$*, if there is some $B \in \mathcal{S}$ with $A_1 \subseteq B$ and $A_2 \subseteq \overline{B}$.
3. (A_1, A_2) is *potentially $\mathcal{S}$-complete*, if it is both $\mathcal{S}$-hard and potentially in $\mathcal{S}$.

If A_2 is the complement of A_1, this definition cuts down to the usual notion of A_1 being $\mathcal{S}$-complete.

Now, for the remainder of this section, let $\mathcal{T} = (T, \tau)$ be an countable T_0 space with countable strong basis, total numbering B of the basis, and numbering x of the space. As is readily verified, the index set of any Lacombe set is r.e., if x is computable. Let us therefore consider the case of nonopen subsets of T.

Theorem 6.3 ([33]) *Let $\mathcal{T}$ be effective and x acceptable. Moreover, let X be a nonopen subset of T such that its complement $\overline{X}$ contains an enumerable dense set. Then $(\overline{K}, K) \leq_1 (\Omega(X), \Omega(\overline{X}))$. In particular, $\Omega(X)$ is productive.*

Here, K denotes the halting set.

Corollary 6.4 ([33]) *Let $\mathcal{T}$ be effective and x acceptable. For any nonopen subset X of T with completely enumerable complement, $(\Omega(X), \Omega(\overline{X})) \equiv_1 (\overline{K}, K)$. In particular, $(\Omega(X), \Omega(\overline{X}))$ is potentially Π^0_1-complete.*

An additional consequence of Theorem 6.3 is a generalization of Rice's Theorem.

Theorem 6.5 ([33]) *Let $\mathcal{T}$ be effective and x acceptable. Moreover, let $\mathcal{T}$ be connected and X be a subset of T. Then X is completely recursive, if and only if either T is empty or the whole space.*

Eršov [11] has shown that Rice's theorem is true for arbitrary precomplete numbered sets. (He considers only total numberings, however the proof remains valid also for partial numberings.) But it is not known, whether acceptable numberings of effective topological spaces are always precomplete.

Let us next consider the problem to decide whether a given point is nonfinite. Let FIN and NFIN, respectively, be the sets of all finite and nonfinite points of T.

Lemma 6.6 ([32]) *Let $\mathcal{T}$ be effective, B negative, and x acceptable. Then* $(\Omega(\text{NFIN}), \Omega(\text{FIN}))$ *is potentially in* Π^0_2.

Note that in [32] a slightly stronger assumption on B was used, but as can be seen from the proof, the above requirement suffices.

As we shall see now, in important cases $\Omega(\text{NFIN})$ is even Π^0_2-complete, if it is not empty. Let to this end, for $y \in T$, $\downarrow y = \{ z \in T \mid z \leq_\tau y \}$.

Theorem 6.7 ([32]) *Let $\mathcal{T}$ be effective and x strongly acceptable. Moreover, let X be a subset of T which contains a nonfinite point y such that $X \cap \downarrow y$ has no finite elements. Then $\Omega(X)$ is Π^0_2-hard.*

It is not clear, how a similar result can be obtained for $(\Omega(X), \Omega(\overline{X}))$.

Corollary 6.8 ([32]) *Let $\mathcal{T}$ be effective and contain a point below which there are no finite points. Moreover, let x be strongly acceptable. Then x cannot be total.*

The assumption is particularly true, when T contains no finite points at all, as in the case of the computable reals. Since the numbering γ of $\mathbb{R}_c$ introduced in Section 4.3 is strongly acceptable—it is even a standard numbering—we have that γ is necessarily partial. It follows that we cannot expect that a constructive metric space has a total strongly acceptable numbering.

As a further consequence of Theorem 6.7 the index set of any nonempty subset of NFIN is Π^0_2-hard, in particular $\Omega(\text{NFIN})$ is Π^0_2-hard. Note that it does not follow from Lemma 6.6 that $\Omega(\text{NFIN}) \in \Pi^0_2$. All we know is that $\Omega(\text{NFIN})$ and $\Omega(\text{FIN})$ are separated by a Π^0_2 set. For a large class of spaces and indexings including the computable real numbers with numbering γ this is however true.

Theorem 6.9 ([32]) *Let $\mathcal{T}$ be effective, constructively complete and contain a nonfinite element. Moreover, let B be negative, $\prec_B$ be r.e. and x be a standard numbering. Then* $\Omega(\text{NFIN})$ *is Π^0_2-complete.*

Since all computable real numbers are nonfinite, this means that $\text{dom}(\gamma)$ is Π^0_2-complete.

7 Degree structures

As we have just seen, numberings of topological spaces can in general not be assumed to be total. In numbering theoretic studies this has always been done so far. We encountered examples where results carry over smoothly. The situation is different, however, in the case of degrees.

Let S be a countable set and $\mathrm{Num}(S)_p$ be the set of all partial numberings of S. The set of all total numberings of S is denoted by $\mathrm{Num}(S)$. Let

$$\deg_p(\nu) = \{\, \kappa \in \mathrm{Num}_p(S) \mid \nu \equiv \kappa \,\}, \quad (\nu \in \mathrm{Num}_p(S)),$$
$$\deg(\nu) = \{\, \kappa \in \mathrm{Num}(S) \mid \nu \equiv \kappa \,\}, \quad (\nu \in \mathrm{Num}(S)),$$

respectively, be the *total* and the *partial degree* of ν. The reduction relation $\leq$ can be lifted to the sets of degrees in the usual way. Thus we obtain the two partial orders

$$\mathcal{L}_p(S) = (\{\, \deg_p(\nu) \mid \nu \in \mathrm{Num}_p(S) \,\}, \leq), \qquad \mathcal{L}(S) = (\{\, \deg(\nu) \mid \nu \in \mathrm{Num}(S) \,\}, \leq).$$

As is well-known, $\mathcal{L}(S)$ is an upper semilattice in which the least upper bound of degrees of ν and κ is induced by the *join* $\nu \oplus \kappa$ defined as follows: for $a \in \omega$,

$$(\nu \oplus \kappa)_{2a} = \begin{cases} \nu_a & \text{if } a \in \mathrm{dom}(\nu), \\ \text{undefined} & \text{otherwise,} \end{cases} \qquad (\nu \oplus \kappa)_{2a+1} = \begin{cases} \kappa_a, & \text{if } a \in \mathrm{dom}(\kappa), \\ \text{undefined} & \text{otherwise.} \end{cases}$$

If S contains more than one element, $\mathcal{L}(S)$ is not a lattice [11].

In the case of $\mathcal{L}_p(S)$ also the greatest lower bound of two degrees exists. It is induced by the *meet* $\nu \sqcap \kappa$ of the numberings ν and κ. For $m, n \in \omega$,

$$(\nu \sqcap \kappa)_{\langle m,n \rangle} = \begin{cases} \nu_m & \text{if } m \in \mathrm{dom}(\nu),\ n \in \mathrm{dom}(\kappa) \text{ and } \nu_m = \kappa_n, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Proposition 7.1 ([2]) *Let S be a countable set. Then the following two statements hold:*

1. *$\mathcal{L}_p(S)$ is a distributive lattice.*
2. *$\mathcal{L}(S)$ is embeddable into $\mathcal{L}_p(S)$ as upper subsemilattice.*

The map $\deg(\nu) \mapsto \deg_p(\nu)$ is an order-isomorphism which preserves finite least upper bounds.

For a total numbering ν the set $\{\, \kappa \in \mathrm{Num}(S) \mid \kappa \leq \nu \,\}$ is countable, as there are only countably many witness functions f. In the case of partial numberings any function that agrees with f, but has a smaller domain of definition may be used as witness function.

Lemma 7.2 ([2]) *For any $\nu \in \mathrm{Num}_p(S)$, $\{\, \kappa \in \mathrm{Num}_p(S) \mid \kappa \leq \nu \,\}$ is uncountable.*

Proof: Consider the *cylindrification* $c(\nu)$ defined by

$$c(\nu)_{\langle i,j \rangle} = \begin{cases} \nu_i & \text{if } i \in \mathrm{dom}(\nu), \\ \text{undefined} & \text{otherwise,} \end{cases} \qquad (i, j \in \omega).$$

Then $c(\nu) \equiv \nu$ and for each $s \in S$, $c(\nu)^{-1}(\{s\})$ is infinite. Let $\kappa \in \mathrm{Num}_p(S)$ with $\kappa^{-1}(\{s\}) \subseteq c(\nu)^{-1}(\{s\})$, for each $s \in S$. Then $\kappa \leq c(\nu)$ via the identity function and there are uncountably many such κ.

A *Friedberg* numbering is a one-to-one numbering. In the case of total numberings they are known to be minimal with respect to the reducibility preorder.

Proposition 7.3 ([2]) *Let S be infinite. Then the degree of any partial Friedberg numbering is not minimal in $\mathcal{L}_p(S)$.*

Given a Friedberg numbering ν, by using a similar argument as above combined with a diagonalization construction a Friedberg numbering is produced which is reducible to ν, but not vice versa.

An infinite set has uncountably many pairwise incomparable total Friedberg numberings [11]. For partial Friedberg numberings an improved statement holds.

Proposition 7.4 *Let S be infinite and $\nu \in \mathrm{Num}_p(S)$. Then there are uncountably many pairwise incomparable partial Friedberg numberings reducible to ν.*

By applying the construction used in Proposition 7.3 to any of them and iterating it, one can strengthen that statement.

Corollary 7.5 ([2]) *Let S be infinite. Then there is an infinite descending chain generated by partial Friedberg numberings below every degree in $\mathcal{L}_p(S)$.*

Let us now return to the case of effective T_0 spaces. As a consequence of Lemma 3.10 we have that the degree of a computable numbering consists only of computable numberings. Moreover, join and meet of computable numberings are computable again. Let

$$\mathcal{C}(\mathcal{T}) = \{\, \mathrm{deg}_p(\nu) \mid \nu \in \mathrm{Num}_p(T) \wedge \nu \text{ computable} \,\}.$$

If existing, the acceptable numberings form a single degree which is largest in $\mathcal{C}(\mathcal{T})$. It consists only of such numberings.

Proposition 7.6 ([37]) *Let $\mathcal{T}$ be an effective T_0 space. Then the following statements hold:*

1. *$\mathcal{C}(\mathcal{T})$ is an ideal in $\mathcal{L}_p$. In particular, $\mathcal{C}(\mathcal{T})$ is a distributive lattice.*
2. *If $\mathcal{T}$ has a computable numbering and an r.e. strong inclusion, $\mathcal{C}(\mathcal{T})$ has a greatest element.*

Statement (2) is a result of Proposition 3.8.

As we have seen, partial numberings are much different from total ones. The above results mainly follow by the fact that given a numbering ν and a reducibility function f, one can define new numberings by using for each $s \in S$ only some of the indices in $f^{-1}(\nu^{-1}(\{s\}))$. This can no longer be done in case of strong reducibility. For $\nu \in \mathrm{Num}_p(S)$ let

$$\mathrm{deg}_s(\nu) = \{\, \kappa \in \mathrm{Num}_p(S) \mid \kappa \equiv_s \nu \,\}$$

be the *strong degree* of ν. As with the other reducibility relations, strong reducibility can be lifted to a partial order on the collection of all strong degrees:

$$\mathcal{L}_s(S) = (\{\, \mathrm{deg}_s(\nu) \mid \nu \in \mathrm{Num}_p(S) \,\}, \leq_s).$$

Lemma 7.7 ([37]) *$\mathcal{L}_s(S)$ is an upper semilattice with the least upper bound of the degrees of numberings ν and κ being given by the degree of $\nu \oplus \kappa$.*

With respect to strong reducibility partial numberings behave more like total ones. In order to see this we will apply Eršov's completion construction for total numberings to partial numberings. It allows to set up a correspondence between partial numberings and complete total numberings. Let to this end

$$\widehat{S} = \{\, \{s\} \mid s \in S \,\} \cup \{\emptyset\}$$

and $\iota\colon S \to \widehat{S}$ with $\iota(s) = \{s\}$ be the corresponding canonical embedding. Moreover, let $u \in P^{(1)}$ be a universal function and $u' \in R^{(1)}$ defined by $u'(c) = \mu a : u(a) = c$ be its right inverse. For $\nu \in \mathrm{Num}_p(S)$ set

$$\hat{\nu}_a = \begin{cases} \{\nu_{u(a)}\} & \text{if } a \in u^{-1}(\mathrm{dom}(\nu)), \\ \emptyset & \text{otherwise,} \end{cases} \qquad (a \in \omega).$$

Lemma 7.8 ([37]) *Let $\nu \in \mathrm{Num}_p(S)$. Then the following two statements hold:*

1. *$\hat{\nu}$ is a complete total numbering of $\widehat{S}$ with special element $\emptyset$.*
2. *$\hat{\nu}_{u'(c)} = \iota(v_c)$, for all $c \in \mathrm{dom}(\nu)$.*

Thus, ι as well as its partial inverse are effective maps. Moreover, the corestriction of $\hat{\nu}$ to $\iota(S)$ is equivalent to $\iota \circ \nu$. Now, let

$$\mathcal{C}_\emptyset(\widehat{S}) = (\{\deg(\rho) \mid \rho \in \mathrm{Num}(\widehat{S}) \wedge \rho \text{ complete with special element } \emptyset\}, \leq).$$

Note that the join of complete total numberings need not be complete again. Nevertheless, we have the following result.

Theorem 7.9 ([37]) *Let S be a countable set. Then the following statements hold:*

1. *$\mathcal{L}_s(S)$ is a monotone retract of $\mathcal{L}(\widehat{S})$.*
2. *$\mathcal{C}_\emptyset(\widehat{S})$ is an upper semilattice.*
3. *The two upper semilattices $\mathcal{L}_s(S)$ and $\mathcal{C}_\emptyset(\widehat{S})$ are isomorphic.*

8 Totalization

In a certain sense, working with total numberings is much easier than working with partial numberings. Remember e.g. the discussion about the difficulty of classifying decision problems in the case of partially numbered sets. Therefore, the question comes up, whether given a partially indexed set (S, ν), there is a total indexing $\overline{\nu}$ of S which extends ν. In applications one is of course not interested in just a total extension of a given partial indexing: important properties such as acceptability should be preserved. As we have already seen, important spaces like the computable reals have canonical strongly acceptable indexings, but do *not* have a total indexing of this kind.

In the last section we presented a construction that allows totalizing a given partial numbering. To this end we had to enlarge the given set, but just by one point. The new total indexing was even complete and both the canonical embedding of the old space into the new one as well as its partial inverse were effective. But does this construction preserve properties like acceptability?

Let $\mathcal{T}$ be an effective T_0 space again with numbering x and let $\widehat{T}$ and $\hat{x}$, respectively, be obtained from T and x as in the last section. Set

$$\widehat{B}_0 = \widehat{T}, \quad \widehat{B}_{n+1} = \iota(B_n),$$

let $\widehat{\mathcal{B}}$ be the collection of the sets $\widehat{B}_a$ $(a \in \omega)$, and define

$$m \prec_{\widehat{B}} n \Leftrightarrow n = 0 \vee [m \neq 0 \wedge n \neq 0 \wedge m - 1 \prec_B n - 1].$$

Then $\widehat{\mathcal{B}}$ is a strong basis of a T_0 topology $\hat{\tau}$ on $\widehat{T}$. Moreover, $\widehat{\mathcal{T}} = (\widehat{T}, \hat{\tau}, \widehat{\mathcal{B}}, \widehat{B}, \prec_{\widehat{B}}, \hat{x})$ is effective.

We will show now that in general, $\hat{x}$ is not computable, even if x is. Assume to this end that both x and $\hat{x}$ are computable. Then there is some r.e. set $\widehat{L}$ such that $\hat{x} \in \widehat{B}_n$, exactly if $\langle i, n \rangle \in \widehat{L}$, for $i, n \in \omega$. Let $A = \{ j \in \omega \mid (\exists n > 0)\langle j, n \rangle \in \widehat{L} \}$. Then A is r.e. Moreover, we have that

$$i \in A \Leftrightarrow \hat{x}_i \in \iota(T) \Leftrightarrow i \in u^{-1}(\mathrm{dom}(x)).$$

Hence, $\mathrm{dom}(x) = u(A)$, which shows that $\mathrm{dom}(x)$ is r.e. This is not true in general: in case of the computable real numbers with its canonical indexing γ we know by Theorem 6.9 that $\mathrm{dom}(\gamma) \in \Pi^0_2$.

In the above construction we enlarged T by one more finite element. As we have seen in Corollary 6.8, an effective space can only have a strongly acceptable numbering, if below each of its nonfinite points there is a finite point. By definition the neighbourhood filter of a finite point $z \in T$ is generated by a single basic open set, say B_n. Then $B_n = \{ y \in T \mid z \leq_\tau y \}$. We will now consider spaces such that all basic open sets are pointed in a certain way.

For $n \in \omega$, let

$$\mathrm{hl}(B_n) = \bigcap \{ B_m \mid n \prec_B m \}.$$

Definition 8.1 Let $\mathcal{T} = (T, \tau)$ be a countable T_0-space with a countable strong basis $\mathcal{B}$, and let x and B be numberings of T and $\mathcal{B}$, respectively. We say that $\mathcal{T}$ is *effectively pointed*, if there is a function $\mathrm{pd} \in P^{(1)}$ such that for all $n \in \mathrm{dom}(B)$ for which B_n is not empty, $\mathrm{pd}(n){\downarrow} \in \mathrm{dom}(x)$, $x_{\mathrm{pd}(n)} \in \mathrm{hl}(B_n)$ and $x_{\mathrm{pd}(n)} \leq_\tau z$, for all $z \in B_n$.

Note that

$$B_n \subseteq \{ z \in T \mid x_{pd(n)} \leq_\tau z \} \subseteq \mathrm{hl}(B_n).$$

Clearly, constructive A-spaces and domains are effectively pointed. Conversely, effectively pointed spaces have typical properties of domains.

Lemma 8.2 ([32]) *Let $\mathcal{T}$ be effective and effectively pointed, and let x be computable. Moreover, let $y \in T$ and $n \in \omega$. Then the following hold:*

1. *$\mathcal{T}$ is recursively separable with dense base $\{ x_a \mid a \in \mathrm{range}(pd) \}$.*
2. *The set $\{ x_{pd(a)} \mid y \in B_a \}$ is directed and y is its least upper bound.*
3. *If m is an index of a converging normed recursive enumeration of basic open sets, then the enumeration converges to the least upper bound of $\left(x_{pd(\varphi_m(a))}\right)_{a \in \omega}$.*
4. *If y is finite, then $y \in \{ x_a \mid a \in \mathrm{range}(pd) \}$.*
5. *If $x_{pd(n)}$ is finite, then $\mathrm{hl}(B_n) = \{ z \in T \mid x_{pd(n)} \leq_\tau z \}$.*

For effectively pointed spaces the existence of a total acceptable numbering has very strong structural consequences.

Theorem 8.3 ([35]) *Let $\mathcal{T}$ be effective and effectively pointed with computable numbering x. Then T has a total acceptable numbering, if and only if $\mathcal{T}$ is constructively d-complete and τ is the Scott topology. The numbering can be chosen as complete, exactly if T has also a smallest element. It agrees with the special element of the numbering.*

It is easy to see that a constructive A-space is effectively complete, just if it is constructively d-complete and its topology is the Scott topology. Thus, we have that exactly the effectively complete constructive A-spaces have total acceptable numberings. Effectively complete constructive A-spaces that have a least element coincide with the bounded-complete constructive domains.

The last theorem tells us that we can totalize an acceptable numbering x such that the resulting numbering $\hat{x}$ is acceptable as well, if we embed the given space into a constructive domain.

Theorem 8.4 ([35]) *Let $\mathcal{T}$ be effective with r.e. strong inclusion so that all basic open sets are not empty. Moreover, let x be acceptable. Then there is an algebraic constructive domain $\widehat{T}$ with a total acceptable complete numbering $\hat{x}$ and an effectively homeomorphic embedding $F\colon T \to \hat{T}$ such that both F and its partial inverse are effective and $F(T)$ is a dense subset of $\hat{T}$.*

The construction is as follows: for $m, n \in \omega$ let

$$m \preccurlyeq n \Leftrightarrow n = 0 \vee m = n \vee [m \neq 0 \wedge n \neq 0 \wedge m-1 \prec_B n-1].$$

Then the relation $\preccurlyeq$ is obviously r.e., reflexive, and transitive with 0 as greatest element. Define $\widehat{T}$ to be the set of all r.e. filters of $\preccurlyeq$, i.e., the collection of all nonempty r.e. subsets of ω which are upwards closed with respect to $\preccurlyeq$ and which with any two elements m and n contain an element a such that $a \preccurlyeq m$, n, and order it by set inclusion. Then $(\widehat{T}, \subseteq)$ is a partial order with the filter $\{0\}$ as smallest element.

Let $s \in R^{(1)}$ be an enumeration of all indices i such that W_i is not empty. Since $\preccurlyeq$ is r.e., a function $h \in R^{(1)}$ can be constructed so that $\varphi_{h(i)}$ tries to enumerate a longest descending chain in $W_{s(i)}$. Set

$$\hat{x}_i = \{\, m \in \omega \mid (\exists a) \varphi_{h(i)}(a) \preccurlyeq m \,\}.$$

Then $\hat{x}$ is a total numbering of $\widehat{T}$. As is readily verified, $(\hat{T}, \subseteq)$ is constructively d-complete with respect to this numbering. Note that the least upper bound of a directed enumerable subset of $\hat{T}$ is the union of all filters in this set.

For $n \in \omega$, let $\hat{z}_n = \{\, m \in \omega \mid n \preccurlyeq m \,\}$. Then the collection of these elements is an algebraic basis of $\widehat{T}$. An easy verification shows that $\widehat{T}$ is an algebraic constructive domain.

The Scott topology on $\hat{T}$ has as canonical basis the collection of all sets

$$\hat{B}_n = \{\, \hat{y} \in \hat{T} \mid \hat{z}_n \subseteq \hat{y} \,\}.$$

It is not hard now to show that $\hat{x}$ is acceptable. Moreover, it is complete with the least element of $\widehat{T}$ as special element.

Since for any $y \in T$ the set of all B_n such that $y \in B_n$ is a strong base of the neighbourhood filter of y, it follows that $\{\, n+1 \mid y \in B_n \,\} \cup \{0\}$ is a filter with respect to $\preccurlyeq$. Define $F\colon T \to \hat{T}$ by

$$F(y) = \{\, n+1 \mid y \in B_n \,\} \cup \{0\}.$$

Then F is one-to-one and both F and its partial inverse are effective. Note that $x_i \in B_n$ if and only if $F(x_i) \in \widehat{B}_{n+1}$. Hence, both F and F^{-1} are effectively continuous. Since the basic open sets B_n are not empty, we also have that $F(T)$ is dense in $\widehat{T}$.

Note that if the strong inclusion relation $\prec_B$ is recursive, it is decidable in m and n whether B_m and B_n are disjoint, and $\mathcal{B}$ is effectively closed under nonempty finite meets, then $\widehat{T}$ is even a constructive Scott domain. Here effective closure under nonempty finite meets means that there is a function $d \in P^{(2)}$ such that for nondisjoint B_m and B_n, $d(m,n)\downarrow$, $B_{d(m,n)} = B_m \cap B_n$ and $d(m,n) \prec_B m, n$.

Furthermore in Theorem 8.4, if $\mathcal{T}$ is also constructively complete, we obtain with Proposition 3.13 that $F(T)$ is even effectively dense in $\widehat{T}$, which means that given $\widehat{B}_n$ we can effectively find a point $\hat{x}_i \in F(T) \cap \widehat{B}_n$.

Embeddings of topological spaces in domains have been much studied in the literature. Mostly one is interested in the case that the embedded space coincides with the subspace of maximal domain elements. Let $\mathrm{Max}(T)$ be the set of maximal points of T with respect to the specialization order.

Proposition 8.5 ([35]) *Let $\mathcal{T}$ be effective and constructively complete so that all basic open sets are nonempty and the strong inclusion relation is r.e. Moreover, let x be acceptable, $\widehat{T}$ be the algebraic constructive domain constructed in Theorem 8.4, and $F\colon T \to \widehat{T}$ be the embedding of T in $\widehat{T}$. Then $F(\mathrm{Max}(T)) = \mathrm{Max}(\widehat{T})$.*

As is well-known, for T_1 spaces the specialization order coincides with the identity. Thus, if in addition $\mathcal{T}$ is T_1, then $\mathcal{T}$ is effectively homeomorphic to the subspace of maximal elements of $\widehat{T}$.

In Section 4.4 we have already pointed out that the constructive completeness notion depends on the choice of the strong inclusion relation. The same is of course true for the construction of the domain $\widehat{T}$.

Consider again the space $\mathbb{R}_c$ of all computable real numbers with the induced Euclidean topology. As a constructive metric space $\mathbb{R}_c$ is constructively complete. Thus, it is effectively homeomorphic to the subspace of maximal elements of the domain constructed as above with respect to the strong basis and the strong inclusion relation introduced for constructive metric spaces in Section 4.3.

If the collection $\mathcal{I}$ of all open intervals with rational endpoints is chosen as topological basis together with the strong inclusion relation $\prec_I$ defined in Section 4.4, $\mathbb{R}_c$ is no longer constructively complete. Nevertheless, it is effectively homeomorphic to the subspace of maximal elements of the domain constructed as above with respect to $\mathcal{I}$ and $\prec_I$.

Now, instead of $\prec_I$ consider the following relation

$$a \prec'_I b \Leftrightarrow \nu_{\pi_1(f(b))} \le \nu_{\pi_1(f(a))} \wedge \nu_{\pi_2(f(a))} \le \nu_{\pi_2(f(b))}.$$

Then also $\prec'_I$ is an r.e. strong inclusion with respect to which $\mathcal{I}$ is a strong basis. Again, $\mathbb{R}_c$ is not constructively complete. But in this case, space $\widehat{T}$ is a constructive Scott domain, in which the embedded computable reals are no longer maximal.

To see this, let $q \in \mathbb{Q}$. By definition, $F(q) = \{\, a+1 \mid \nu_{\pi_1(f(a))} < q < \nu_{\pi_2(f(a))} \,\} \cup \{0\}$. As is easily verified the following sets

$$J^1_q = \{\, a+1 \mid \nu_{\pi_1(f(a))} < q \le \nu_{\pi_2(f(a))} \,\} \cup \{0\}$$
$$J^2_q = \{\, a+1 \mid \nu_{\pi_1(f(a))} \le q < \nu_{\pi_2(f(a))} \,\} \cup \{0\}$$

are also filters with respect to the preorder $\preccurlyeq$ derived from $\prec'_I$. Neither is J^1_q contained in J^2_q nor is J^2_q contained in J^1_q, but $F(q)$ is properly contained in both of them.

9 Final remarks

In this paper we have presented a general approach to effectively given topological spaces. A class of spaces with suitable indexings was defined that includes a large variety of well-known examples and is closed under important constructions of new spaces from given ones, thus showing the robustness of the concept.

Numberings of the kind considered here exist under very natural conditions usually satisfied in applications. With respect to these numberings important topological operations like limit passing become effective. Canonical bases of the neighbourhood filters of the points can uniformly be enumerated. Typically such numberings are only partially defined. We have seen that this is so by necessity. We have also seen that dealing with partial numberings is more difficult than doing so with total numberings. Notions introduced for the latter do not always carry over easily. There are refinements of different strength. Therefore, the question of whether such numberings can be totalized was considered.

As has already been said when introducing effective spaces, in this approach we assume that the spaces we are interested in come equipped with a notion of computable point. We consider only the subspaces induced by these points. Doing so allows to represent these points by natural numbers and to use classical computability theory over the natural numbers.

Other approaches to computable topology like Weihrauch's Type Two Theory of Effectivity (TTE) [45] do not restrict themselves to subspaces of computable elements. The elements can now no longer be represented by natural numbers. In the TTE approach elements of Baire space, i.e. functions on the natural numbers are used instead. The relationship between this approach and the one presented here has been studied in [36].

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