

Effectivity and Effective Continuity of Multifunctions

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1. Introduction

Multifunctions have been used with great success in applied mathematics and logic (in particular modal and temporal logic).

2. Framework

We are working in the framework of effectively given topological T_0 spaces. Let

- ▶ T be a topological T_0 space with countable basis

$$\mathcal{B} = \{B_0, B_1, \dots\}.$$

We think of the basic open sets as elementary predicates that are easy to encode.

In general it is difficult to deal with set inclusion in an effective framework. In most cases we can use a stronger relation on the codes of basic open sets instead.

Definition

A transitive relation \prec_B on ω is a *strong inclusion*, if for all $m, n \in \omega$

$$m \prec_B n \Rightarrow B_m \subseteq B_n.$$

Assume further that

- ▶ \prec_B is r.e.
- ▶ \mathcal{B} is a *strong basis*, i.e., the property of being a base holds with respect to \prec_B instead of \subseteq .

Since T is T_0 , every point $y \in T$ is uniquely determined by its neighbourhood filter

$$\mathcal{N}(y) = [\{ B_n \mid y \in B_n \}].$$

- ▶ Assume that for all $y \in T$, $\{ n \in \omega \mid y \in B_n \}$ is r.e.

This gives rise to an indexing x of T with the following properties

- ▶ Given i , we can enumerate all n with $x_i \in B_n$, uniformly in i .
- ▶ Given an index of a recursive sequence $a_0 \succ_B a_1 \succ_B \cdots$, if $\{B_{a_\nu} \mid \nu \in \omega\}$ is a strong base of $\mathcal{N}(y)$, for some $y \in T$, we can compute an index of y .

Note. In general, x is only a partial map, i.e., not every number is an index of a point of T .

3. How to encode sets.

Since there are uncountably many subsets X of T , not every subset will be encodeable.

a) Encode an enumeration procedure for X .

- ▶ Enumerate all indices of all points of X

Definition

X is *completely enumerable* (c.e.) if there is some r.e. set A such that for all $i \in \text{dom}(x)$

$$i \in A \Leftrightarrow x_i \in X.$$

Every r.e. index of A is called *c.e. index* of X .

- ▶ Enumerate at least one index for every point in X .

Definition

X is *enumerable* if there is some r.e. set $A \subseteq \text{dom}(x)$ with

$$X = \{x_i \mid i \in A\}.$$

Every r.e. index of A is called *enumeration index* of X .

By enumerating all elements of a set we have effective access to all of them. This limits of course the kind of sets we can deal with in a computable way. Sometimes it is sufficient to enumerate a generating part of X or certain properties of the points of X .

b) Density indices

Definition

X is *effectively dense* if there is some enumerable dense subset Y of X . Any enumeration index of Y is called *density index* of X .

Lemma

Given an enumeration index of X , we can compute a density index of X . The converse is not true, in general.

c) Covering indices

Definition

X is *effectively covered* if the set $\{n \mid B_n \cap X \neq \emptyset\}$ is r.e. Every r.e. index of this set is called *covering index* of X .

Lemma

*Given a density index of X , we can compute a covering index of X .
The converse is not true, in general.*

d) Finite covering indices

Let D be a canonical indexing of all finite subsets of ω .

Definition

X has a *computable finite cover* if the set

$$\{i \mid X \subseteq \bigcup \{B_a \mid a \in D_i\}\}$$

is r.e. Every r.e. index of this set is called *finite covering index* of X .

A finite covering index codes a procedure that enumerates *all* finite covers of X . Therefore one needs a stronger property than just *compactness* to compute a finite covering index from a covering index of X .

Definition

X is *strongly effectively compact* if there is a total computable function g so that for every covering index i of X , $g(i)$ is an r.e. index of

$$\{ n \mid D_n \subseteq W_i \wedge X \subseteq \bigcup \{ B_a \mid a \in D_n \} \}$$

.

Lemma

Let X be strongly effectively compact. Then, given a covering index of X , we can compute a finite covering index of X .

In important special cases density indices can be obtained from covering indices and enumeration indices from density indices.

Definition

T is *constructively complete* if for every computable sequence $a_0 \succ_B a_1 \succ_B \cdots$, there is a point $y \in T$ such that $\{B_{a_\nu} \mid \nu \in \omega\}$ is a strong base of $\mathcal{N}(y)$.

Proposition

Let T be constructively complete and $X \subseteq T$ be closed. Then, given a covering index of X , we can compute a density index of X .

Proposition

Let T consist of the computable points of an effectively given continuous poset. Then, given a density index of X , we can compute an enumeration index of X .

Note that in general X is not uniquely determined by the indices considered: different sets can have the same index. We think of X as being given by other means. We generate only certain properties of it.

4. Effective maps

Usually a map is defined extensionally using set theory. In recursive mathematics an extensionally given map can be described intensionally

$$\begin{array}{ccc} y & \xrightarrow{F} & F(y) \\ \uparrow & & \uparrow \\ i & \xrightarrow{f} & f(i). \end{array}$$

The function f computes a name of $F(y)$ from the code of a procedure enumerating properties of the argument y .

The map F is called *effective* in this case.

In our case $F(x_i)$ is a set which needs not be uniquely determined by the index $f(i)$. As just said, we think of $F(x_i)$ as given by other (e.g. classical) means. The value $f(i)$ codes useful information about $F(x_i)$.

Question: Is every effective multifunction continuous?

5. Continuity of multifunctions

There are several continuity notions for multifunctions.

- ▶ Lower semicontinuity (lsc)

Definition

$F : T \rightrightarrows T'$ is lsc \Leftrightarrow

$$\begin{aligned} & (\forall y \in T)(\forall U \in \tau') [F(y) \cap U \neq \emptyset \\ & \quad \Rightarrow (\exists V \in \mathcal{N}(y))(\forall z \in V) F(z) \cap U \neq \emptyset] \\ & \quad \Leftrightarrow (\forall U \in \tau') F^{-}(U) \in \tau, \end{aligned}$$

where $F^{-}(U) = \{z \in T \mid F(z) \cap U \neq \emptyset\}$.

In a second countable space every open set is a countable union of basic open sets. This gives rise to the following notion of being effectively open.

Definition

$O \in \tau$ is *Lacombe open* if there is some r.e. set A so that

$$O = \bigcup \{ B_a \mid a \in A \}.$$

Any r.e. index i of A is called *Lacombe index* of O . We write $O = L_i^\tau$.

Definition

$F : T \rightrightarrows T'$ is *effectively lsc* if there is some total computable function g so that

$$F^-(B'_n) = L_{g(n)}^\tau.$$

► Upper semicontinuity (usc)

Definition

$$\begin{aligned} F : T \rightrightarrows T' \text{ is usc} &\Leftrightarrow \\ (\forall y \in T)(\forall U \in \tau') [F(y) \subseteq U] & \\ \Rightarrow (\exists V \in \mathcal{N}(y))(\forall z \in V) F(z) \subseteq U & \\ \Leftrightarrow (\forall U \in \tau') F^+(U) \in \tau, & \end{aligned}$$

where $F^+(U) = \{z \in T \mid F(z) \subseteq U\}$.

We know that U is the union over a not necessarily effective list of basic open sets. If we want to proceed as above in defining effective usc we would have to require that we can compute a Lacombe index for $F^+(U)$ relative to any listing of basic open sets whose union is U . We do not want to do this. Instead we restrict ourselves to maps $F : T \rightrightarrows T'$ with $F(z)$ being compact, for every $z \in T$. In this case it is sufficient to consider only finite unions U of basic open sets.

Let

$$U_n = \bigcup \{ B_a \mid a \in D_n \}.$$

Definition

A compact-valued map $F : T \rightrightarrows T'$ is *effectively usc* if there is some total computable function g so that

$$F^+(U'_n) = L_{g(n)}^\tau.$$

Note that $\{F^-(B'_n) \mid n \in \omega\}$ and $\{F^+(U'_n) \mid n \in \omega\}$, respectively, are subbases of topologies $F^-(\tau')$ and $F^+(\tau')$. Moreover,

- ▶ F is effectively lsc $\Leftrightarrow F^-(\tau') \subseteq_e \tau$
- ▶ F is effectively usc $\Leftrightarrow F^+(\tau') \subseteq_e \tau$

where a topology η on T with subbasis $\{C_n \mid n \in \omega\}$ is *effectively coarser* than τ ($\eta \subseteq_e \tau$) if there is a total computable function g with

$$C_n = L_{g(n)}^\tau,$$

for $n \in \omega$.

A condition that forces a topology η on T to be effectively coarser than the given topology τ :

Definition

Let η be a topology on T with subbasis $\{C_n \mid n \in \omega\}$. A pair of (s, r) of computable functions is a *noninclusion realizer* of τ with respect to η if

- ▶ $x_i \in C_m \Rightarrow x_i \in M_{s(i,m)} \subseteq C_m$
- ▶ If, in addition, $B_n \not\subseteq C_m$, then $x_{r(i,n,m)} \in B_n \setminus M_{s(i,m)}$.

Here, M is a numbering of the c.e. sets.

Definition

- ▶ T is *recursively separable* if it has an enumerable dense subset.
- ▶ η is a *Mal'cev topology* on T if it has a subbasis of c.e. sets.

Theorem

Let T be recursively separable and η be a Mal'cev topology on T . Then, if τ has a noninclusion realizer with respect to η , then $\eta \subseteq_e \tau$.

The converse implication holds as well.

Lemma

Let τ have a noninclusion realizer with respect to τ . Then, if $\eta \subseteq_e \tau$ then τ has a noninclusion realizer with respect to η .

Note.

- ▶ If $F : T \rightrightarrows T'$ is effective with respect to covering indices, then $F^-(B'_n)$ is c.e., uniformly in n . Thus $F^-(\tau')$ is a Mal'cev topology on T .
- ▶ If $F : T \rightrightarrows T'$ is compact-valued and effective with respect to finite covering indices, then $F^+(U'_n)$ is c.e., uniformly in n . Thus $F^+(\tau')$ is a Mal'cev topology on T .

Thus, in order to obtain that F is effectively lsc or usc, respectively, we have to ensure that τ has a noninclusion realizer with respect to $F^-(\tau')$ and $F^+(\tau')$.

We will now study important special cases and see when such realizers exist.

- ▶ Effectively given continuous posets with the Scott topology

Proposition

Let T consist of the computable points of an effectively given continuous poset. The Scott topology σ has a noninclusion realizer with respect to any Mal'cev topology on T .

Theorem

Let T consist of the computable points of an effectively given continuous poset and $F : T \rightrightarrows T'$. Then

- ▶ *F is effective with respect to covering indices iff F is effectively lsc.*
- ▶ *If F is compact-valued. Then F is effective with respect to finite covering indices iff F is effectively usc.*

- ▶ Recursively separable recursive metric spaces

Definition

- ▶ $X \subseteq T$ has an *effective complement exhaustion* if the set

$$\{n \mid B_n \cap X = \emptyset\}$$

is r.e. Any r.e. index of this set is called *complement exhaustion index* of X .

- ▶ A pair $\langle i, j \rangle$ is a *strong covering index* of X if i is a covering and j a complement exhaustion index of X .

Theorem

Let T, T' be recursively separable recursive metric spaces and $F : T \rightrightarrows T'$ be effective with respect to strong covering indices. Then τ has a noninclusion realizer with respect to $F^-(\tau')$ and hence F is effectively lsc.

Definition

The pair $\langle i, j \rangle$ is a *strong finite covering index* of X if i is a finite covering and j a covering index of X .

Proposition

Let T, T' be recursively separable recursive metric spaces and $F : T \rightrightarrows T'$ be compact-valued. If F is effective with respect to strong finite covering indices, then τ has a noninclusion realizer with respect to $F^+(\tau')$ and hence F is effectively usc.

As we will see now, F is also effectively lsc in this case.

Lemma

Let T be a recursively separable recursive metric space and $X \subseteq T$ be compact. If X has a finite covering index i , then X also has an effective complement exhaustion, uniformly in i .

Theorem

Let T, T' be recursively separable recursive metric spaces and $F : T \rightrightarrows T'$ be compact-valued. If F is effective with respect to strong finite covering indices, then F is both effectively lsc and usc, i.e., F is effectively continuous.

Remember that if a set X is strongly effectively compact, then from a covering index of X we can compute a finite covering index of X .

Corollary

Let T, T' be recursively separable recursive metric spaces and $F : T \rightrightarrows T'$ be effective with respect to covering indices. If $F(x_i)$ is strongly effectively compact, uniformly in i , then F is effectively continuous.

- ▶ Outer semicontinuity (osc)

Definition

Let $F : T \rightrightarrows T'$.

- ▶ $\limsup_{y \rightarrow \bar{y}} F(y) = \{ u \in T' \mid (\exists y^\nu \rightarrow \bar{y})(\exists u^\nu \rightarrow u) u^\nu \in F(y^\nu) \}$
- ▶ F is *osc* at \bar{y} if $\limsup_{y \rightarrow \bar{y}} F(y) \subseteq F(\bar{y})$.

Theorem

- ▶ F *osc everywhere* \Leftrightarrow
 $(\forall \bar{y} \in T)(\forall u \notin F(\bar{y}))(\exists W \in \mathcal{N}(u))(\exists V \in \mathcal{N}(\bar{y}))F(V) \cap W = \emptyset$.
- ▶ F *osc everywhere* \Leftrightarrow $\text{graph}(F)$ *closed in* $T \times T'$.

Definition

F is *effectively osc* if there are computable maps h, k such that

$$(\forall i \in \text{dom}(x))(\forall j \in \text{dom}(x'))[x'_j \notin F(x_i) \Rightarrow x'_j \in B'_{h(i,j)} \wedge x_i \in B_{k(i,j)} \wedge F(B_{k(i,j)}) \cap B'_{h(i,j)} = \emptyset].$$

Theorem

F is *effectively osc* iff $\text{graph}(F)$ is *Lacombe closed*, i.e., $\text{graph}(F)^c$ is *Lacombe open*.

Theorem

Let $F : T \rightrightarrows T'$ be effective with respect to density indices and F^c with $F^c(y) = F(y)^c$ be effective with respect c.e. indices. Then F is *effectively osc*.

Remember that from a density index of X we can compute a covering index of X . Thus,

- $F : T \rightrightarrows T'$ effective with respect to density indices
- $\Rightarrow F$ effective with respect to covering indices
- $\Rightarrow F$ effectively lsc,

if T consists of the computable points of an effectively given continuous poset.

Moreover, in this case:

Theorem (Rice/Shapiro)

X is c.e. iff X is Lacombe open.

Corollary

Let T consist of the computable points of an effectively given continuous poset. Let $F : T \rightrightarrows T'$ be effective with respect to both density and Lacombe-closure indices. Then F is effectively lsc as well as osc.

Note that for closed $X \subseteq T$,

X has effective complement exhaustion $\Rightarrow X^c$ is c.e.

Moreover, a c.e. index of X^c can be computed from a complement exhaustion index of X . Thus, we have for $F : T \rightrightarrows T'$,

F effective with respect to both density and exhaustion indices

$\Rightarrow F$ effective with respect to strong covering indices

$\Rightarrow F$ effectively lsc,

if T, T' are recursively separable recursive metric spaces.

Corollary

Let T, T' be recursively separable recursive metric spaces and let $F : T \rightrightarrows T'$ be closed-valued. Then, if F is effective with respect to both density and complement exhaustion indices, then F is both effectively lsc and osc.

If, in addition, $F(x_i)$ is strongly effectively compact, uniformly in i , then F is also effectively usc.