Domains under bitopological glasses

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1. A short domain-theory fresh-up

Definition

- A poset \((D, \sqsubseteq)\) with least element \(\bot\) is called \text{domain} if every directed set \(S \subseteq D\) has a least upper bound \(\bigcup S\) in \(D\).
- \(D\) is \text{bounded-complete} if every bounded subset has a least upper bound in \(D\).
- Let \(x, y \in D\). Then \(x\) approximates \(y\) \((x \ll y)\) if for all directed sets \(S \subseteq D\),
  \[ y \sqsubseteq \bigcup S \quad \Rightarrow \quad (\exists s \in S)x \sqsubseteq s. \]
- Element \(x \in D\) is \text{compact} if \(x \ll x\).
- \(B \subseteq D\) is a \text{basis} for \(D\) if for every \(x \in D\),
  \(B_x = \{ z \in B \mid z \ll x \}\) is directed with least upper bound \(x\).
Remark

\[ B \text{ basis for } D \quad \Rightarrow \quad B \supseteq K_D(= \{ \text{compact elements} \}) \]

Definition

- \( D \) is \textit{continuous} if \( D \) has a basis.
- \( D \) is \textit{algebraic} if \( K_D \) is a basis for \( D \).

Remark

\textit{In a continuous domain} \( \ll \) \textit{has the interpolation property:}

\[ M \subseteq_f D \& M \ll y \quad \Rightarrow \quad (\exists z \in B)M \ll z \ll y. \]
Let $D$ be a continuous domain with basis $B$. For $x \in D$ and $z \in B$ set
\begin{itemize}
  \item $\uparrow x = \{ y \in D \mid x \sqsubseteq y \}$,
  \item $\uparrow z = \{ y \in D \mid z \ll y \}$.
\end{itemize}

**Definition**
\begin{itemize}
  \item The **Scott topology** $\sigma$ on $D$ is generated by the sets $\uparrow z$ with $z \in B$.
  \item The **lower topology** $\omega$ on $D$ has all principal filters $\uparrow x$ with $x \in D$ as a subbasis for the closed sets.
  \item The **Lawson topology** $\lambda$ on $D$ is the join of both the Scott and the lower topology.
\end{itemize}
For a topology $\tau$ let $\leq_\tau$ denote its specialization order.

Remark

- $\leq_\sigma = \subseteq$,
- $\leq_\omega = \subseteq^{-1}$.
- $(D, \lambda)$ is Hausdorff.
2. A theorem and its appropriate generalization

**Theorem**

A countably based continuous domain with its Lawson topology is completely metrizable.

In order to get rid of the countability assumption one could try to prove:

**Claim**

A continuous domain with its Lawson topology is completely uniformizable.

But is this the generalization one is really looking for?

In applications one is mainly interested in the Scott topology. Therefore, a much more informative generalization would be
Claim

For every continuous domain $D$ a quasi-uniformity $\mathcal{U}$ can be given such that

- $\tau_\mathcal{U} = \sigma$
- $\tau_{\mathcal{U}^{-1}} = \omega$

$(D,\mathcal{U}^*)$ is complete, i.e. $(D,\mathcal{U})$ is bicomplete.

Here,

- $\tau_\mathcal{U}$ denotes the topology induced by $\mathcal{U}$,
- $\mathcal{U}^{-1}$ is the converse of $\mathcal{U}$ and
- $\mathcal{U}^*$ is the uniformity generated by $\mathcal{U}$.

The advantage of this generalization is that in the countably based case one would automatically obtain a quasi-metric the topology of which is compatible with the Scott topology. In certain cases one would even obtain a partial metric. This is what one is really looking for in applications.
3. The algebraic case

Let $D$ be an algebraic domain. Then $K_D$, the set of its compact elements is a basis of $D$ and the collection of all principal filters $\uparrow z$, for $z \in K_D$, is a base of the Scott topology on $D$.

For $z \in K_D$ set

$$P_z = \{ (x, y) \in K^2 \mid z \sqsubseteq x \Rightarrow z \sqsubseteq y \}.$$  

Then $\{ P_z \mid z \in K_D \}$ is a subbasis of a quasi-uniformity $\mathcal{P}$ on $D$.

Note that

$$P_z^{-1} = \{ (y, x) \in K^2 \mid z \not\sqsubseteq y \Rightarrow z \not\sqsubseteq x \}$$
Therefore, we have for $x, y \in D$ and $z \in K_D$ that

$$P_z[x] = \{ y \mid (x, y) \in P_z \} = \begin{cases} \uparrow z & \text{if } x \in \uparrow z, \\ D & \text{otherwise,} \end{cases}$$

and

$$P_z^{-1}[y] = \begin{cases} D \setminus \uparrow z & \text{if } y \notin \uparrow z, \\ D & \text{otherwise.} \end{cases}$$
Proposition

Let $D$ be an algebraic domain. Then the following hold:

1. $\tau_\mathcal{P} = \sigma$.
2. $\tau_{\mathcal{P}^{-1}} = \omega$.
3. $\mathcal{P}$ is the coarsest quasi-uniformity on $D$ compatible with $\sigma$.
4. $\mathcal{P}$ is totally bounded.
4. The general (continuous) case

Let $D$ be continuous domain with basis $B$. For $z, z' \in B$ with $z \ll z'$ define

$$K_{z,z'} = \{ (x, y) \in K^2 \mid z' \ll x \Rightarrow z \ll y \}.$$ 

Then

$$\{ K_{z,z'} \mid z, z' \in B \text{ with } z' \ll z \}$$

is a subbasis of the Künzi-Brüummer quasi-uniformity $\mathcal{K}$ on $D$.

**Lemma**

*If $D$ is algebraic, then $\mathcal{K} = \mathcal{P}$.***
Proposition

Let $D$ be a continuous domain. Then the following hold:

1. $\tau_K = \sigma$.
2. $\tau_{K^{-1}} = \omega$.
3. $K$ is the coarsest quasi-uniformity on $D$ compatible with $\sigma$.
4. $K$ is totally bounded.

The first three statements are a special case of a more general result of J. Lawson.
5. Bicompleteness

**Definition**
A domain is **coherent** if the intersection of two Scott-compact saturated sets is again Scott-compact.

**Theorem**
Let $D$ be a coherent continuous domain. Then $(D, \mathcal{K})$ is bicomplete.

**Corollary**
$(D, \tau_{\mathcal{K}^*})$ is compact.
Remark

- As we have seen above, $\tau_{K^*}$ coincides with the Lawson topology on $D$.
- It is well known that

$$ (D, \lambda) \text{ is compact} \iff D \text{ is coherent.} $$

Thus, in the above theorem we cannot dispense with coherence.
Remember the initial

Claim
For every continuous domain $D$ there is a quasi-uniformity $\mathcal{U}$ such that

\begin{itemize}
  \item $\tau_\mathcal{U} = \sigma$
  \item $\tau_{\mathcal{U}^{-1}} = \omega$
  \item $(D, \mathcal{U})$ is bicomplete.
\end{itemize}

We do not know whether this claim is true. If so, we would have

\begin{itemize}
  \item $\mathcal{K} \not\subseteq \mathcal{U}$ and
  \item $\mathcal{U}$ is not totally bounded.
\end{itemize}
6. The Urysohn construction

Let $D$ be a continuous domain with Basis $B$ and $\mathbb{D}$ be the set of dyadic rationals in the interval $[0, 1]$.

By repeated interpolation we can, for any pair $(z, z') \in B^2$ with $z \ll z'$, construct a family $\langle e_p \rangle_{p \in \mathbb{D}}$ such that

- $e_0 = z'$, $e_1 = z$
- $e_q \ll e_p$, for all $p, q, \in \mathbb{D}$ with $p < q$.

Define $f_{z'}: D \to [0, 1]$ by

$$f_{z'}^z(x) = \inf \{ p \in \mathbb{D} | e_p \ll x \}.$$

**Lemma**

- $f_{z'}^z$ is $\sigma$-upper and $\omega$-lower semicontinuous.
- $f_{z'}^z(\uparrow z') = \{0\}$.
- $f_{z'}^z(D \setminus \uparrow z) = \{1\}$. 
For $z, z' \in B$ with $z \ll z'$ and $m > 0$ let

$$U_{z,z',m} = \{ (x, y) \in D^2 \mid f_z^{z'}(y) - f_z^{z'}(x) < 2^{-m} \}.$$ 

Proposition

The collection

$$\{ U_{z,z',m} \mid m > 0 \text{ and } z, z' \in B \text{ with } z \ll z' \}$$

is a subbasis of a quasi-uniformity $\mathcal{U}$ on $D$ such that

$$\mathcal{U} = \mathcal{K}.$$
Assume now that $D$ is countably based. Let

$$(z_0, z'_0), (z_1, z'_1), \ldots$$

be an enumeration of all pairs $(z, z') \in B^2$ with $z \ll z'$. Set

$$f_i = f^{z'_i}_{z_i}, \quad U_{i,m} = U_{z_i,z'_i,m}$$

and define for $x, y \in D$

$$\delta(x, y) = \sum_{i=0}^{\infty} 2^{-(i+1)} \max\{0, f_i(y) - f_i(x)\}.$$ 

**Lemma**

- $\delta$ is a quasi-metric on $D$.
- $\mathcal{V}_\delta = \mathcal{U}$

Here, $\mathcal{V}_\delta$ is the quasi-uniformity induced by $\delta$. 

Definition
A quasi-metric $d$ on a continuous domain $D$ is weakly weighted if there is measurement $|\cdot| : D \to [0, 1]^\text{op}$ such that for $x, y \in D$,

$$x \sqsubseteq y \Rightarrow |y| + d(y, x) \leq |x|.$$ 

Conjecture (Smyth)
For any countably based continuous domain $D$ there is a weakly weighted quasi-metric $d$ with measurement $|\cdot|$ such that

- $|x| = 0 \iff x$ is constructively maximal,
- $d$ induces the Scott topology,
- $d^*$ induces the Lawson topology.

Here, $d^*$ is the metric associated with $d$. 
Definition
Let $D$ be a domain. We say $x, y \in D$ lie apart from each other and write $x \sharp y$, if both can be separated by disjoint Scott open sets.

Let $\langle e^i_p \rangle_{p \in D}$ be the family of interpolating basic elements constructed with respect to the pair $(z_i, z'_i)$. Set

$$|x| = 1 - \sum_{i=0}^{\infty} 2^{-(i+1)} \left[ \sup \{ p \in D \mid x \sharp e^i_p \} + (1 - f_i(x)) \right].$$

Then $| \cdot |$ is a measurement so that

$$|x| = 0 \iff x \text{ is constructively maximal}.$$ 

Moreover, it turns $\delta$ into a weakly weighted quasi-metric.
Theorem

Let $D$ be a countably based continuous domain. Then $\delta$ is a weakly weighted quasi-metric with measurement $|\cdot|$ such that

- $|x| = 0 \iff x$ is constructively maximal.
- $\delta$ induces the Scott topology,
- $\delta^*$ induces the Lawson topology.
7. The Smyth partial metric

**Definition**
A *partial metric* on a set $X$ is a map $p: X \times X \to [0, \infty)$ satisfying

1. $p(x, y) = p(y, x),$
2. $[p(x, y) = p(x, x) = p(y, y)] \Rightarrow x = y,$
3. $p(x, z) \leq p(x, y) + p(y, z) - p(y, y),$
4. $p(x, x) \leq p(x, y).$

It is well known that the notions $T_0$ weighted quasi-metric and partial metric are equivalent, via the assignment

$$p(x, y) = w(x) + d(x, y)$$

and its inverse

$$w(x) = p(x, x), d(x, y) = p(x, y) - p(x, x).$$

The topology induced by a partial metric is the one induced by the associated quasi-metric.
Let $D$ be a countably based continuous domain. For $x, y \in D$ set

$$\rho(x, y) = 1 - \sum_{i=0}^{\infty} 2^{-(i+1)} [\sup \{ p \mid e_p \# x, y \} + \sup \{ 1 - p \mid e_p \ll x, y \}].$$

**Theorem (Smyth)**

The distance function $\rho$ is a partial metric which induces the Scott topology.
Let \( D \) be a continuous domain with (not necessarily countable) basis \( B \). For \( z, z' \in B \) with \( z \ll z' \) set

\[
S_{z,z'} = \{ (x, y) \in D^2 \mid [z' \ll x \Rightarrow z \ll y] \land [z\# x \Rightarrow z'\# y] \}.
\]

Then the collection of all such relations \( S_{z,z'} \) is a subbasis of a quasi-uniformity \( S \) on \( D \).

**Proposition**

\( \mathcal{K} \subseteq S \).

**Corollary**

\( \sigma \subseteq \tau_S, \quad \omega \subseteq \tau_{S^{-1}}. \)
Proposition

$\tau_S = \sigma$.

Remark

I do not know whether

- $\tau_{S^{-1}} = \omega$,  
- $(D, S)$ is bicomplete?

Proposition

If $B$ is countable then

$$S = \mathcal{V}_\rho$$

where $\mathcal{V}_\rho$ is the quasi-uniformity induced by the Smyth partial metric.