Finitely Generated Rank-Ordered Sets as a Model for Type: Type *

Dieter Spreen
Fachbereich Mathematik, Theoretische Informatik
Universität–Gesamthochschule Siegen
D–57068 Siegen, Germany
Email: spreen@informatik.uni-siegen.de

Abstract

The collection of isomorphism classes of finitely generated rank-ordered sets is shown to be a finitely generated rank-ordered set again. This is used to construct a model of the simply typed lambda calculus extended by the assumption Type: Type.

Beside this, the structure of rank-ordered sets is studied. They can be represented as inverse limits of ω-cochains of substructures, each being a retract of the following. The category of such limits is equivalent to the category of rank-ordered sets.

Introduction

There has been a controversial discussion among type theoreticians whether, or not, one should accept that the collection of all types is itself a type (*cf. e.g.* [3]). In the recent systems by Martin-Löf, in Girard's systems F and F_ω, and Coquand-Huet's Calculus of Construction this assumption is carefully avoided, whereas in programming languages like those studied by Burstall and Lampson [5], or by Cardelli [6], it is explicitly used.

Models of such calculi with the axiom Type: Type have been constructed using finitary retractions and/or projections over certain domains [2, 11]. In the present paper we present a model based on rank-ordered sets for a very simple type system, the simply typed λ-calculus, extended by the axiom Type: Type.

A rank-ordered set is a nonempty set with a family of commuting projection functions that satisfy some additional requirements. The projections assign a canonical sequence of approximations to each element of the set. By considering for any two points the length of the longest common subsequence, a distance can be defined that turns the rank-ordered set into a complete ultrametric space. Moreover, letting one element be smaller then another element if the first is obtained from the second by an application of one of the projections, introduces an order relation with which the rank-ordered set is a bounded-complete algebraic directed-complete partial order. If the set is finitely generated, which means that each projection has a finite range, it is even an dI-domain. The metric topology turns out to be just the Lawson topology on this domain.

Rank-ordered sets have been introduced by Bruce and Mitchell [4] in an attempt to extend to higher types the model constructions of Amadio [1] and Cardone [7] for polymorphic λ-calculus with recursive types and subtyping. With rank-preserving functions as morphisms they form a cartesian closed category.

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Rank-preserving functions are such that each approximation of a function value depends at most on approximations of the argument of the same rank. Here, the rank of an element is the minimum of the indices of the projections under which the element is fixed.

Each rank-ordered set can be represented as an inverse limit of the ω-cochain of the ranges of its projection functions. By mapping every rank-ordered set to the range of its ith projection one obtains a family of projection functions that nearly generates a rank-ordered structure on the collection of all rank-ordered sets: One condition which requires that the zeroth projection maps all elements onto a distinguished element is satisfied only up to isomorphism. Thus, by identifying isomorphic rank-ordered sets the collection of all rank-ordered sets can be made into a rank-ordered class. If one restricts oneself to finitely generated rank-ordered sets, the construction can be done in such a way that one obtains a finitely generates rank-ordered set again.

The paper is divided into two parts. In the first part the model is presented and in the second part the structure of rank-ordered sets is studied in more detail. We organize the matter as follows:

Section 1 contains the definitions of rank-ordered sets and rank-preserving functions. Moreover, it is shown that rank-ordered can be obtained as inverse limits of ω-cochains of substructures, each being a retract of the following, and that every such limit is a rank-ordered set.

In Section 2 projection functions on the collection of isomorphism classes of rank-ordered sets are defined which turn it into a rank-ordered class and/or a finitely generated rank-ordered set, in case that only finitely generated rank-ordered sets are considered.

Section 3 concludes the first part of the paper. The type system we are interested in, *i.e.*, the simply typed λ-calculus extended by the axiom Type: Type, is presented and it is verified that by interpreting types as isomorphism classes of finitely generated rank-ordered sets one obtains a model of it.

In Section 4 it is shown that on every rank-ordered set both a partial order and an ultrametric can be defined in a canonical way which turn it into a bounded-complete algebraic directed-complete partial order and a complete ultrametric space, respectively.

Finally, in Section 5, the representation of rank-ordered sets as inverse limits of cochains of sets is studied. Via this representation both the Scott topology and the metric topology on rank-ordered sets can be obtained in a canonical way from the discrete topology on the components of the cochain. The category of these inverse limits is a Cartesian closed full subcategory of the category of rank-ordered sets. Both categories turn out to be equivalent. As a consequence a representation of the space of rank-preserving functions between two rank-ordered sets follows which allows to consider a rank-preserving function as a sequence of maps that operate locally on the components of the cochains representing the rank-ordered sets.

1 Rank-Ordered Sets

It is widely known that every Scott domain D can be represented as the inverse limit of an ω-cochain $(D_i)_{i\in\omega}$ of finite subdomains [13]. Since the subdomains D_i are closed under the operation of taking existing least upper bounds, any element x of D has a best approximation $\bigsqcup\{ z \in D_i \mid z \sqsubseteq x \}$ in each of them. Abstracting from such properties one obtains the notion of a rank-ordered set [4].

Definition 1.1 Let K be a set and $([\cdot]_i)_{i\in\omega}$ be a family of maps $[\cdot]_i\colon K \to K$. $(K, ([\cdot]_i)_{i\in\omega})$ is *rank-ordered* if the following three conditions are satisfied:

1. For all $x \in K$, $[[x]_i]_j = [x]_{\min\{i,j\}}$.

2. For all x, $y \in K$, $[x]_0 = [y]_0$. We write $\perp_K$ for $[x]_0$.

3. For any sequence $(x_i)_{i \in \omega}$ from K with $x_i = [x_{i+1}]_i$, for $i \in \omega$, there is a unique x in K such that for all $i \in \omega$, $x_i = [x]_i$.

If only conditions (1) and (3) hold, we say that $(K, ([\cdot]_i)_{i \in \omega})$ is *pre-rank-ordered*. For $i \in \omega$, let $K_i = \{\, x \in K \mid x = [x]_i \,\}$. Then K is *finitely generated*, if every set K_i is finite.

Bruce and Mitchell use rank-preserving maps as morphisms between rank-ordered sets.

Definition 1.2 Let K and L be rank-ordered sets. A function $f: K \to L$ is *rank-preserving* if for all $x \in K$ and all i, $j \in \omega$ with $j \geq i$,

$$[f(x)]_i^L = [f([x]_j^K)]_i^L.$$

It is easily verified that a function $f: K \to L$ is rank-preserving if and only if for all x, $y \in K$ and all $i \in \omega$,

$$[x]_i^K = [y]_i^K \Rightarrow [f(x)]_i^L = [f(y)]_i^L.$$

As has been shown by Bruce and Mitchell, the collection **RS** of rank-ordered sets with rank-preserving functions is a cartesian closed category. Products are formed as cartesian products $K \times L$ with projections $[\cdot]_i^\times$ given by

$$[(x, y)]_i^\times = ([x]_i^K, [y]_i^L).$$

Exponentials are the sets $[K \Rightarrow L]$ of all rank-preserving functions $f: K \to L$, where the projections are defined by

$$[f]_i^\to(x) = [f(x)]_i^L.$$

Obviously, $(K \times L)_i = K_i \times L_i$ and $[K \Rightarrow L]_i = \{\, f: K \to L \mid (\forall x \in K) f(x) = [f([x]_i^K)]_i^L \,\}$, from which it follows that $K \times L$ and $[K \Rightarrow L]$ are finitely generated, if K and L are.

As we shall see now, rank-ordered sets can be obtained as inverse limits of ω-cochains of sets each of which is a retract of the following.

Theorem 1.3 *Let M be the inverse limit of the cochain $(M^i, p_i)_{i \in \omega}$, where the M^n are nonempty sets such that there is a map $e_n: M^n \to M^{n+1}$ with $p_n \circ e_n = \mathrm{id}_{M^n}$. Then M is a pre-rank-ordered set. If M^0 is a singleton set, it is even rank-ordered. Moreover, if, in addition, all sets M^n are finite, then M is finitely generated.*

Proof: Let $e_{nm} = e_{m-1} \circ \cdots \circ e_n$ and $p_{mn} = p_n \circ \cdots \circ p_{m-1}$, for $n < m$, and $e_{nn} = p_{nn} = \mathrm{id}_{M^n}$. For $x \in M^n$ set

$$\mathrm{in}_n(x)(m) = \begin{cases} p_{nm}(x) & \text{if } m < n, \\ e_{nm}(x) & \text{otherwise.} \end{cases}$$

As it is readily verified, $p_m(\mathrm{in}_n(x)(m+1)) = \mathrm{in}_n(x)(m)$. Thus $\mathrm{in}_n(x) \in M$. Now, for $t \in M$ and $i \in \omega$ set

$$[t]_i^{li} = \mathrm{in}_i(t(i)).$$

Then $[t]_i^{li} \in M$. Moreover,

$$[t]_i^{li}(m) = \begin{cases} t(m) & \text{if } m \leq i, \\ e_{im}(t(i)) & \text{otherwise.} \end{cases}$$

It remains to verify conditions 1.1(1) and (3). For condition (1) let $i,\ j \in \omega$. Then we have that

$$\begin{aligned}
[[t]_i^{li}]_j^{li}(m) &= \begin{cases} [t]_i^{li}(m) & \text{if } m \leq j, \\ e_{jm}([t]_i^{li}(j)) & \text{otherwise} \end{cases} \\
&= \begin{cases} t(m) & \text{if } m \leq \min\{i,j\}, \\ e_{im}(t(i)) & \text{if } i < m \leq j, \\ e_{jm}(t(j)) & \text{if } j < m \text{ and } j \leq i, \\ e_{jm}(e_{ij}(t(i))) & \text{if } i < j < m \end{cases} \quad = [t]_{\min\{i,j\}}^{li}.
\end{aligned}$$

For condition (3), finally, let $(t_i)_{i\in\omega}$ be a sequence in M with $t_i = [t_{i+1}]_i^{li}$, for $i \in \omega$. Then

$$t_i(m) = \begin{cases} t_{i+1}(m) & \text{if } m \leq i, \\ e_{im}(t_{i+1}(i)) & \text{otherwise.} \end{cases}$$

Set $t(m) = t_m(m)$. It follows that

$$p_m(t(m+1)) = p_m(t_{m+1}(m+1)) = t_{m+1}(m) = t_m(m) = t(m),$$

which means that $t \in M$. Moreover,

$$[t]_i^{li}(m) = \begin{cases} t(m) & \text{if } m \leq i, \\ e_{im}(t(i)) & \text{otherwise} \end{cases} = \begin{cases} t_{i+1}(m) & \text{if } m \leq i, \\ e_{im}(t_{i+1}(i)) & \text{otherwise} \end{cases} = t_i(m).$$

If M^0 is a singleton set, say $M^0 = \{\bot\}$, set $\bot_M = \mathrm{in}_0(\bot)$. Then we have that

$$[t]_0^{li} = (e_{0m}(t(0)))_{m\in\omega} = (e_{0m}(\bot))_{m\in\omega} = \mathrm{in}_0(\bot) = \bot_M.$$

Obviously $M_i = \mathrm{in}_i(M^i)$. Therefore M is finitely generated, if the sets M^i are finite.

Theorem 1.4 *Let K be a rank-ordered set and $i \in \omega$. Then the following statements hold:*

1. $K_i \subseteq K_{i+1}$.
2. $[\cdot]_i^K \restriction K_{i+1}\colon K_{i+1} \to K_i$, *onto, such that* $[x]_i^K = x$, *for* $x \in K_i$.
3. K *is isomorphic to the inverse limit of the cochain* $(K_i, [\cdot]_i^K \restriction K_{i+1})_{i\in\omega}$.

Proof: The first two statements are easy to verify. For the third statement let $\vec{K}$ be the inverse limit of the cochain $(K_i, [\cdot]_i^K \restriction K_{i+1})_{i\in\omega}$ and let $t \in \vec{K}$. Then $[t(i+1)]_i^K = t(i)$. Hence there is a unique element $z_t \in K$ with $[z_t]_i^K = t(i)$. Set $g(t) = z_t$. Then $g\colon \vec{K} \to K$. Obviously, if for $n \in \omega$, $t^n \in \vec{K}$ is defined by $t^n(i) = t(i)$, if $i < n$, and $t^n(i) = t(n)$, otherwise, then $z_{t^n} = t(n)$. Hence, we have for $m,\ n \in \omega$ with $n \geq m$ that

$$[g(t)]_m^K = t(m) = [g(t^n)]_m^K = [g([t]_n^{li})]_m^K,$$

which shows that $g \in [\vec{K} \Rightarrow K]$.

Define $f\colon K \to \vec{K}$ by $f(x)(i) = [x]_i^K$. Then it follows for $m,\ n \in \omega$ with $n \geq m$ that

$$[f(x)]_m^{li}(i) = \begin{cases} [x]_m^K & \text{if } i \geq m, \\ [x]_i^K & \text{otherwise} \end{cases} = \begin{cases} [[x]_n^K]_m^K & \text{if } i \geq m, \\ [[x]_n^K]_i^K & \text{otherwise} \end{cases} = [f([x]_n^K)]_m^{li}(i).$$

Thus $f \in [K \Rightarrow \vec{K}]$. Moreover, we have that $g(f(x)) = g(([x]_i^K)_{i\in\omega}) = x$ and $f(g(t)) = f(z_t) = ([z_t]_i^K)_{i\in\omega} = t$.

2 Rank-Ordering the Rank-Ordered Sets

Let $\mathcal{RS}$ be the class of all rank-ordered sets and $\mathcal{FRS}$ be the subclass of all finitely generated rank-ordered sets. An obvious way to define projection functions on $\mathcal{RS}$ is by setting

$$[K]_i^{\mathcal{RS}} = K_i.$$

But then the second condition in 1.1 is satisfied only up to isomorphism. Therefore we consider $\mathcal{RS}/_{\cong}$ instead of $\mathcal{RS}$. Here, $K \cong L$ means that the two rank-ordered sets K and L are isomorphic. Let $\langle K \rangle$ denote the isomorphism class of K. Then we set

$$[\langle K \rangle]_i^{\mathcal{RS}} = \langle K_i \rangle.$$

Note that this definition is independent of the choice of the representative K, as $K \cong L$ implies that $K_i \cong L_i$.

In order to see this, let $f \in [K \Rightarrow L]$ and $g \in [L \Rightarrow K]$ with $g \circ f = \mathrm{id}_K$ and $f \circ g = \mathrm{id}_L$ and set $\bar{f} = [f]_i^{\rightarrow} \restriction K_i$ and $\bar{g} = [g]_i^{\rightarrow} \restriction L_i$. Then $\bar{f} \in [K_i \Rightarrow L_i]$ and $\bar{g} \in [L_i \Rightarrow K_i]$ with $\bar{g} \circ \bar{f} = \mathrm{id}_{K_i}$ and $\bar{f} \circ \bar{g} = \mathrm{id}_{L_i}$. Since every set is up to isomorphism the inverse limit of a constant cochain, it follows from Theorem 1.3 that $\mathcal{RS}/_{\cong}$ is not a set.

Theorem 2.1 *$(\mathcal{RS}/_{\cong}, ([\cdot]_i^{\mathcal{RS}})_{i\in\omega})$ is a rank-ordered class.*

Proof: The first two conditions in 1.1 are readily verified. For condition (3) let $(K^i)_{i\in\omega}$ be a sequence of rank-ordered sets such that $\langle K^i \rangle = [\langle K^{i+1} \rangle]_i^{\mathcal{RS}}$, which means that $K^i \cong K_i^{i+1}$. Let $q_i \in [K_i^{i+1} \Rightarrow K^i]$ be this isomorphism. For $x \in K^{i+1}$ and $y \in K^i$ set

$$p_i(x) = q_i([x]_i^{K^{i+1}}) \quad \text{and} \quad e_i(y) = q_i^{-1}(y).$$

Since $e_i(y) \in K_i^{i+1}$, we obtain that $p_i(e_i(y)) = q_i([q_i^{-1}(y)]_i^{K^{i+1}}) = q_i(q_i^{-1}(y)) = y$. Let $\vec{K}$ be the inverse limit of the cochain $(K^i, p_i)_{i\in\omega}$. Then $\vec{K}$ is a rank-ordered set, by Theorem 1.3. Because $\vec{K}_i$ is the set of all elements $\mathrm{in}_i(t(i))$, for $t \in \vec{K}$, and the maps p_j are onto, it follows that $\vec{K}_i$ is the set of all sequences

$$\begin{pmatrix} e_{in}(y) & (n \geq i) \\ p_{in}(y) & (n < i) \end{pmatrix}_{n\in\omega}.$$

We will now show that $\vec{K}_i \cong K^i$. As $[\mathrm{in}_i(y)]_i^{li} = \mathrm{in}_i(\mathrm{in}_i(y)(i)) = \mathrm{in}_i(y)$, for $y \in K^i$, we have that $\mathrm{in}_i \colon K^i \to \vec{K}_i$. Moreover, for $m,\, n \in \omega$ with $m \geq n$, it is

$$\begin{aligned}
[\mathrm{in}_i([y]_m^{K^i})]_n^{li} &= \mathrm{in}_n(\mathrm{in}_i([y]_m^{K^i})(n)) \\
&= \begin{cases} \mathrm{in}_n(e_{in}([y]_m^{K^i})) & \text{if } n \geq i, \\ \mathrm{in}_n(p_{in}([y]_m^{K^i})) & \text{otherwise} \end{cases} \\
&= \begin{cases} \mathrm{in}_n([e_{in}([y]_m^{K^i})]_n^{K^n}) & \text{if } n \geq i, \\ \mathrm{in}_n(q_n([\cdots[q_{i-1}([[y]_m^{K^i}]_{i-1}^{K^i})]_{i-2}^{K^{i-1}}\cdots]_n^{K^{n+1}})) & \text{otherwise} \end{cases} \\
&= \begin{cases} \mathrm{in}_n([e_{in}(y)]_n^{K^n}) & \text{if } n \geq i, \\ \mathrm{in}_n(q_n([\cdots[q_{i-1}([y]_{i-1}^{K^i})]_{i-2}^{K^{i-1}}\cdots]_n^{K^{n+1}})) & \text{if } n < i \\ & \text{and } i-1 \leq m, \\ \mathrm{in}_n(q_n([\cdots[q_{m-1}([q_m \circ \cdots \circ q_{i-1}([y]_m^{K^i})]_{m-1}^{K^m})]_{m-2}^{K^{m-1}}\cdots]_n^{K^{n+1}})) & \text{otherwise} \end{cases} \\
&= \begin{cases} \mathrm{in}_n(e_{in}(y)) & \text{if } n \geq i, \\ \mathrm{in}_n(p_{in}(y)) & \text{if } n < i \\ & \text{and } i-1 \leq m, \\ \mathrm{in}_n(q_n([\cdots[q_{m-1}([q_m \circ \cdots \circ q_{i-1}(y)]_{m-1}^{K^m})]_{m-2}^{K^{m-1}}\cdots]_n^{K^{n+1}})) & \text{otherwise} \end{cases}
\end{aligned}$$

$$= \begin{cases} \mathrm{in}_n(e_{in}(y)) & \text{if } n \geq i, \\ \mathrm{in}_n(p_{in}(y)) & \text{if } n < i \text{ and } i-1 \leq m, \\ \mathrm{in}_n(q_n([\cdots[q_{i-1}([y]^{K^i}_{i-1})]^{K^{i-1}}_{i-2}\cdots]^{K^{n+1}}_n)) & \text{otherwise} \end{cases}$$

$$= \begin{cases} \mathrm{in}_n(e_{in}(y)) & \text{if } n \geq i, \\ \mathrm{in}_n(p_{in}(y)) & \text{otherwise} \end{cases}$$

$$= [\mathrm{in}_i(y)]^{li}_n.$$

Here, we used that the functions involved are rank-preserving and that $e_{in}(y) \in K^n_n$, for $y \in K^i$. It follows that $\mathrm{in}_i \in [K^i \Rightarrow \vec{K}_i]$.

Next, set $\mathrm{pr}_i(t) = t(i)$, for $t \in \vec{K}_i$. Then $\mathrm{pr}_i \colon \vec{K}_i \to K^i$. Furthermore, for m, n as above we have that $[t]^{li}_m = t$, if $m \geq i$. Hence $[\mathrm{pr}_i([t]^{li}_m)]^{K^i}_n = [\mathrm{pr}_i(t)]^{K^i}_n$, in this case. If $m < i$ it is

$$[\mathrm{pr}_i([t]^{li}_m)]^{K^i}_n = [[t]^{li}_m(i)]^{K^i}_n = [[t(i)]^{K^i}_m]^{K^i}_n = [t(i)]^{K^i}_n = [\mathrm{pr}_i(t)]^{K^i}_n.$$

It follows that $\mathrm{pr}_i \in [\vec{K}_i \Rightarrow K^i]$. Obviously $\mathrm{pr}_i(\mathrm{in}_i(x)) = x$, for $x \in K^i$. Moreover, $\mathrm{in}_i(\mathrm{pr}_i(t)) = \mathrm{in}_i(t(i)) = [t]^{li}_i = t$, for $t \in \vec{K}_i$. This shows that pr_i is the desired isomorphism.

As a consequence we obtain for $i \in \omega$ that

$$[\langle \vec{K} \rangle]^{\mathcal{RS}}_i = \langle \vec{K}_i \rangle = \langle K^i \rangle.$$

We will now restrict ourselves to finitely generated rank-ordered sets. As follows from Theorems 1.3 and 1.4, they are up to isomorphism the sets that can be obtained as inverse limits of ω-cochains of finite sets, in which each set is a retract of the following. This implies that the collection of isomorphism classes of finitely generated rank-ordered sets is a set again.

In what follows, for $K \in \mathcal{FRS}$, $\langle K \rangle$ will denote the isomorphism class of K in $\mathcal{FRS}$. We set

$$[\langle K \rangle]^{\mathcal{FRS}}_i = \langle K_{\max\{\, j \leq i \mid |K_j| \leq i+1 \,\}} \rangle.$$

Here $|K_j|$ denotes the cardinality of K_j.

Theorem 2.2 *$(\mathcal{FRS}/_{\cong}, ([\cdot]^{\mathcal{FRS}}_i)_{i\in\omega})$ is a finitely generated rank-ordered set.*

Proof: Obviously, $\mathcal{FRS}/_{\cong}$ is finitely generated. We now verify the conditions in 1.1.

For condition (1) let i, $j \in \omega$, $k = \max\{\, c \leq i \mid |K_c| \leq i+1 \,\}$, and $r = \max\{\, s \leq j \mid |(K_k)_s| \leq j+1 \,\}$. Then $[[\langle K \rangle]^{\mathcal{FRS}}_i]^{\mathcal{FRS}}_j = \langle (K_k)_r \rangle$. If $i \leq j$, it follows that $k \leq i \leq j$ and $|K_k| \leq i+1 \leq j+1$. Hence, $(K_k)_j = K_k$. As a consequence we obtain that $r = j$. Thus we have that $[[\langle K \rangle]^{\mathcal{FRS}}_i]^{\mathcal{FRS}}_j = \langle (K_k)_j \rangle = \langle K_k \rangle = [\langle K \rangle]^{\mathcal{FRS}}_i$.

Let us now consider the case that $j < i$. If $r > k$, then $k \leq j$ and $(K_k)_r = K_k$. It follows that $|K_k| \leq j+1$. Hence, $k = \max\{\, s \leq j \mid |K_s| \leq j+1 \,\}$. This shows that $[[\langle K \rangle]^{\mathcal{FRS}}_i]^{\mathcal{FRS}}_j = \langle (K_k)_r \rangle = \langle K_k \rangle = [\langle K \rangle]^{\mathcal{FRS}}_j$.

On the other hand, if $r \leq k$, then $(K_k)_r = K_r$. Thus $|K_r| \leq j+1$. Now, let $r' \leq j$ with $|K_{r'}| \leq j+1$. Then $r' < i$ and $|K_{r'}| < i+1$. Hence $r' \leq k$. It follows that $K_{r'} = (K_k)_{r'}$. Since $r = \max\{\, s \leq j \mid |(K_k)_s| \leq j+1 \,\}$, this implies that $r' \leq r$. Thus $r = \max\{\, e \leq j \mid |K_e| \leq j+1 \,\}$, which means that $[[\langle K \rangle]^{\mathcal{FRS}}_i]^{\mathcal{FRS}}_j = \langle (K_k)_r \rangle = \langle K_r \rangle = [\langle K \rangle]^{\mathcal{FRS}}_j$.

Condition (2) is obviously satisfied. For (3) let $(K^i)_{i\in\omega}$ be a sequence of finitely generated rank-ordered sets with $\langle K^i \rangle = [\langle K^{i+1} \rangle]^{\mathcal{FRS}}_i$, for $i \in \omega$. Then

$$K^i \cong K^{i+1}_{\max\{\, j \leq i \mid |K^{i+1}_j| \leq i+1 \,\}}.$$

Let $j(i) = \max\{ j \leq i \mid |K_j^{i+1}| \leq i+1 \}$, and let the isomorphism from $K_{j(i)}^{i+1}$ onto K^i be given by q_i. For $x \in K^{i+1}$ and $y \in K^i$ set

$$p_i(x) = q_i([x]_{j(i)}^{K^{i+1}}) \quad \text{and} \quad e_i(y) = q_i^{-1}(y).$$

Since $e_i(y) \in K_{j(i)}^{i+1}$, we obtain that $p_i(e_i(y)) = q_i([e_i(y)]_{j(i)}^{K^{i+1}}) = q_i(q_i^{-1}(y)) = y$. Let $\vec{K}$ be the inverse limit of the cochain $(K^i, p_i)_{i\in\omega}$. Note that $|K^i| = |K_{j(i)}^{i+1}| \leq i+1$. By Theorem 1.3 we therefore have that $\vec{K}$ is a finitely generated rank-ordered set again. From the proof of Theorem 2.1 we know that $\vec{K}_i = K^i$. Thus

$$\max\{ s \leq i \mid |\vec{K}_s| \leq i+1 \} = \max\{ s \leq i \mid |K^s| \leq i+1 \} = i.$$

It follows that

$$[\langle \vec{K} \rangle]_i^{\mathcal{FRS}} = \langle \vec{K}_{\max\{ s \leq i \mid |\vec{K}_s| \leq i+1 \}} \rangle = \langle \vec{K}_i \rangle = \langle K^i \rangle.$$

3 A Rank-Ordered Model of Type: Type

The type system we shall consider in this section is the simply typed λ-calculus (*cf.* [12]) extended by the axiom Type: Type. Its signature $\langle \mathcal{B}, \mathcal{C} \rangle$ consists of a set $\mathcal{B}$ of *base types* and a collection $\mathcal{C}$ of disjoint sets $\mathcal{C}^\sigma$ indexed by type expressions over $\mathcal{B}$; the elements of $\mathcal{C}^\sigma$ are called *constant symbols of type* σ. The set $\mathcal{B}$ contains a special type constant $*$, which is interpreted as the collection of all types. Moreover, $\mathcal{C}^* = \{*\}$.

Using b to stand for base types, the *type expressions* of the calculus are defined by the grammar

$$\sigma = b | \sigma_1 \to \sigma_2.$$

Expressions of the calculus and their types are defined simultaneously via typing assertions

$$\Gamma \triangleright a : \tau,$$

where Γ is a *type assignment* of the form

$$\Gamma = \{ x_1 : \sigma_1, \ldots, x_n : \sigma_n \},$$

with no x_i occurring twice. We set $\mathrm{Var}(\Gamma) = \{x_1, \ldots, x_n\}$ in this case. Note that instead of $\Gamma \cup \{x : \sigma\}$ we also write $\Gamma, x : \sigma$. Intuitively, the assertion $\Gamma \triangleright a : \tau$ says that if the variables $x_1, \ldots, x_n$ have types $\sigma_1, \ldots, \sigma_n$, respectively, then a is a well-formed term of type τ. Typing assertions are defined inductively by the following typing axioms and rules:

$*.1 \quad \emptyset \triangleright c : \sigma, \quad$ for each constant symbol c of type σ,

$*.2 \quad x : \sigma \triangleright x : \sigma,$

$$*.3 \quad \frac{\Gamma \triangleright a : \sigma}{\Gamma, x : \tau \triangleright a : \sigma} \quad (x \notin \mathrm{Var}(\Gamma)),$$

$$*.4 \quad \frac{\Gamma, x : \sigma \triangleright a : \tau}{\Gamma \triangleright (\lambda x : \sigma . a) : \sigma \to \tau},$$

$$*.5 \quad \frac{\Gamma \triangleright a : \sigma \to \tau, \; \Gamma \triangleright b : \sigma}{\Gamma \triangleright ab : \tau}.$$

Note that as a special case of $(*.1)$ we have the axiom $\emptyset \triangleright * : *$.

In the model, which we are going to define now, types are interpreted as finitely generated rank-ordered sets. Let T be a map that assigns a finitely generated rank-ordered set to each base type such that $T(*) = \mathcal{FRS}/\cong$. T is extended to type expressions by setting

$$T(\sigma_1 \to \sigma_2) = [T(\sigma_1) \Rightarrow T(\sigma_2)].$$

Furthermore, let C be a map which associates an element of $T(\sigma)$ with each constant symbol of type σ. In particular $C(*) = \langle \mathcal{FRS}/\cong \rangle$.

An *environment* η is a mapping from variables to the union of all finitely generated rank-ordered sets. If Γ is a type assignment, then η is said to *satisfy* Γ, if $\eta(x) \in T(\sigma)$ for every $x : \sigma \in \Gamma$. For any environment η and $d \in \bigcup \mathcal{FRS}$, $\eta[d/x]$ is the environment mapping y to $\eta(y)$ for y different from x and x to d.

The meaning $[\![\Gamma \triangleright a : \tau]\!]_\eta$ of a well-formed term $\Gamma \triangleright a : \tau$ with respect to an environment η which satisfies Γ is defined by recursion on the derivation of the term:

$$
\begin{aligned}
&(1) \quad [\![\emptyset \triangleright c : \sigma]\!]_\eta = C(c),\\
&(2) \quad [\![x : \sigma \triangleright x : \sigma]\!]_\eta = \eta(x),\\
&(3) \quad [\![\Gamma, x : \sigma \triangleright a : \tau]\!]_\eta = [\![\Gamma \triangleright a : \tau]\!]_\eta,\\
&(4) \quad [\![\Gamma \triangleright (\lambda x : \sigma . a) : \sigma \to \tau]\!]_\eta = \text{the unique } f \in T(\sigma \to \tau) \text{ such that}\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } d \in T(\sigma),\ f(d) = [\![\Gamma, x : \sigma \triangleright a : \tau]\!]_{\eta[d/x]},\\
&(5) \quad [\![\Gamma \triangleright ab : \tau]\!]_\eta = [\![\Gamma \triangleright a : \sigma \to \tau]\!]_\eta([\![\Gamma \triangleright b : \sigma]\!]_\eta).
\end{aligned}
$$

For the justification of clause (4) one shows that for any typing assertion $\Gamma \triangleright a : \tau$ with $\Gamma = \{x_1 : \sigma_1, \ldots, x_n : \sigma_n\}$ and any environment η which satisfies Γ, the function mapping $(d_1, \ldots, d_n) \in T(\sigma_1) \times \cdots \times T(\sigma_n)$ to $[\![\Gamma \triangleright a : \tau]\!]_{\eta[d_1/x_1]\cdots[d_n/x_n]}$ is rank-preserving. In the proof one uses that the category of finitely generated rank-ordered sets with rank-preserving functions is Cartesian closed. It follows that this category defines an environment model of the simply typed λ-calculus in which the axiom $\emptyset \triangleright * : *$ holds.

4 More on Rank-Ordered Sets

In this and the following section we will study the structure of rank-ordered sets in more detail. As we shall see a partial order as well as an ultrametric can be defined on such sets in a canonical way which turn it into a bounded-complete algebraic directed-complete partial order and a complete ultrametric space, respectively.

Let $(D, \sqsubseteq)$ be a partial order with smallest element $\perp$. A subset S of D is called *compatible* if it has an upper bound. S is *directed*, if it is nonempty and every pair of elements in S has an upper bound in S. D is a *directed-complete* partial order (cpo) if every directed subset S of D has a least upper bound $\bigsqcup S$, and D is *bounded-complete* if every compatible subset has a least upper bound.

An element x of a cpo D is *compact* if for any directed subset S of D the relation $x \sqsubseteq \bigsqcup S$ always implies the existence of an element $u \in S$ with $x \sqsubseteq u$. We write D^0 for the set of compact elements of D. If for every $y \in D$ the set $\{ x \in D^0 \mid x \sqsubseteq y \}$ is directed and $y = \bigsqcup \{ x \in D^0 \mid x \sqsubseteq y \}$, the cpo D is said to be *algebraic*.

Definition 4.1 Let D be a cpo.

1. D is a *Scott domain* if it is bounded-complete and algebraic and D^0 is countable.
2. D is a *dI-domain* if it is a Scott domain and the two axioms d and I are satisfied:
   - Axiom d: For all x, y, $z \in D$, if $\{y, z\}$ is compatible then
   $$x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$$
   - Axiom I: For all $x \in D^0$, $\{ y \in D \mid y \sqsubseteq x \}$ is finite.

Note that in a bounded-complete algebraic cpo any two elements x and y have a greatest lower bound $x \sqcap y$.

Definition 4.2 Let D and E be bounded-complete algebraic cpo's. A function $f: D \to E$ is said to be

1. *Scott continuous* if for any directed subset S of D the image set $f(S)$ is also directed and
$$f(\bigsqcup S) = \bigsqcup f(S).$$
2. *stable* if it is Scott continuous and for every compatible pair $\{x, y\}$ of elements of D we have that
$$f(x \sqcap y) = f(x) \sqcap f(y).$$

Obviously, Scott continuous functions are monotone. Moreover, they are exactly the functions that are continuous with respect to the *Scott topology*. For a bounded-complete algebraic cpo D this topology is generated by the sets $O_z = \{\, x \in D \mid z \sqsubseteq x \,\}$ with $z \in D^0$.

As it is well known, the categories of all Scott domains with continuous maps as morphisms and of all dI-domains with stable maps as morphisms are cartesian closed. In both cases products are formed as cartesian products ordered coordinatewise. For Scott domains D and E the function space consists of the set of Scott continuous functions $f: D \to E$ ordered argumentwise and for dI-domains D and E the function space consists of the set of stable functions $f: D \to E$ ordered by the *stable ordering* $\sqsubseteq_s$, where $f \sqsubseteq_s g$ if for all x, $y \in D$
$$x \sqsubseteq y \Rightarrow f(x) = f(y) \sqcap g(x).$$
For an introduction to the theory of domains the reader is referred to [9].

Now, let K be a rank-ordered set. For x, $y \in K$, define
$$x \sqsubseteq y \Leftrightarrow x = y \vee (\exists i \in \omega) x = [y]_i.$$

Proposition 4.3 *Let K be a rank-ordered set. Then $(K, \sqsubseteq, \bot_K)$ is a bounded-complete algebraic cpo that satisfies axioms d and I.*

Proof: Obviously, $\sqsubseteq$ is reflexive and transitive. For antisymmetry let $x = [y]_i$ and $y = [x]_j$ with $i \leq j$. Then $y = [[y]_i]_j = [y]_i = x$. Thus, $(K, \sqsubseteq)$ is a partial order with $\bot_K$ as smallest element.

Claim 1 Let S be a compatible subset of K. Then S is a chain.

We have to show for x, $y \in S$ that either $x \sqsubseteq y$ or $y \sqsubseteq x$. Let $z \in K$ be an upper bound of S. Then x, $y \sqsubseteq z$. If $x = z$ or $y = z$ we are done. Let us therefore assume that $x \neq z$ and $y \neq z$. Then there are i, $j \in \omega$ such that $x = [z]_i$ and $y = [z]_j$. Without restriction let $i \leq j$. It follows that $x = [z]_i = [[z]_j]_i = [y]_i$, which means that $x \sqsubseteq y$

It follows that each directed subset of K is a chain.

Claim 2 Every chain in K has a least upper bound in K,

Let $(x_i)_{i \in \omega}$ be a chain in K. Without restriction we assume that all its elements are pairwise different. Then there are numbers n_i such that $x_i = [x_{i+1}]_{n_i}$, for all $i \in \omega$. Let the n_i be minimal with this property.

Suppose that for some index i, $n_{i+1} \leq n_i$. Then
$$x_i = [x_{i+1}]_{n_i} = [[x_{i+2}]_{n_{i+1}}]_{n_i} = [x_{i+2}]_{n_{i+1}} = x_{i+1},$$

which contradicts our assumption. Thus $n_i < n_{i+1}$, for all $i \in \omega$.

Now, set $t(i) = \min\{\, j \mid i \leq n_j \,\}$ and define

$$z_i = [x_{t(i)}]_i.$$

Then $x_i = z_{n_i}$. Moreover, $z_i = [z_{i+1}]_i$. Let z be the unique element of K with $[z]_i = z_i$, for all $i \in \omega$. It follows that z is an upper bound of the chain (x_i).

Let z' be a further upper bound. Then there are numbers m_i such that $x_i = [z']_{m_i}$. As above the assumption that $m_{i+1} \leq n_i$, for some index i, leads to a contradiction. Therefore $n_i < m_{i+1}$, for all $i \in \omega$. With this we have that

$$[z]_{n_i} = z_{n_i} = x_i = [x_{i+1}]_{n_i} = [[z']_{m_{i+1}}]_{n_i} = [z']_{n_i}.$$

Since the sequence of the n_i is strictly increasing, we obtain that $[z]_j = [z']_j$, for all $j \in \omega$, which implies that $z = z'$. Thus z is the least upper bound of the chain (x_i).

This shows that $(K, \sqsubseteq, \bot_K)$ is a bounded-complete cpo.

Claim 3 $K^0 = \bigcup\{\, K_i \mid i \in \omega \,\}$

Let $x \in K_i$ and z be the least upper bound of a chain $(z_j)_{j\in\omega}$ with $x \sqsubseteq z$. As we have seen above, the z_j can be chosen such that $z_j = [z_{j+1}]_j$. Then $z_j = [z]_j$, for all indices j. If $x = z$ then $z \in K_i$. Hence $z = [z]_i = z_i$, *i. e.*, $x \sqsubseteq z_i$. If $x \neq z$, there is some $n \in \omega$ with $x = [z]_n$. Then $x = [z]_n = z_n$. So, we have $x \sqsubseteq z_n$. This shows that x is compact.

Conversely, let $x \in K$ be compact. Since $\{\, [x]_i \mid i \in \omega \,\}$ is a directed subset with least upper bound x, there is some $i \in \omega$ with $x = [x]_i$. Hence $x \in K_i$.

It follows that for any $x \in K$ the set $\{\, [x]_i \mid i \in \omega \,\}$ is the set of compact elements below x. Since it is obviously directed and x is its least upper bound, we have that $(K, \sqsubseteq, \bot_K)$ is also algebraic.

In Claim 1 we have seen that any two compatible elements are comparable with respect to the partial order. This implies that $(K, \sqsubseteq, \bot_K)$ satisfies axiom d.

Claim 4 Let $x \in K^0$. Then the set $\{\, y \in K \mid y \sqsubseteq x \,\}$ is finite.

Let i be minimal with $x \in K_i$. Moreover, let $y \sqsubseteq x$ and assume that $y \neq x$. Then there exists some number j with $y = [x]_j$. Suppose that $j \geq i$. In this case we have that

$$y = [x]_j = [[x]_i]_j = [x]_i = x,$$

contrary to our choice of y. Thus $j < i$, which implies that there are only finitely many elements of K below x.

It follows that for finitely generated rank-ordered sets K, $(K, \sqsubseteq, \bot_K)$ is a dI-domain. Moreover, since any two compatible elements of a rank-ordered set are comparable with respect to the above defined partial order, we have that every Scott continuous function between such sets is also stable and, as follows from the next result, the stable ordering $\sqsubseteq_s$ on the function space coincides with the pointwise ordering $\sqsubseteq_p$.

Lemma 4.4 *Let f, $g\colon K \to L$ be Scott continuous. Then $f \sqsubseteq_p g$ if and only if $f \sqsubseteq_s g$.*

Proof: Obviously, we only have to show the "only if"-part. Let x, $y \in K$ such that $x \sqsubseteq y$ and suppose that $f \sqsubseteq_p g$. Moreover, without restriction, assume that both $x \neq y$ and $f \neq g$. We have to prove that $f(x) = f(y) \sqcap g(x)$. Note that since $f(x) \sqsubseteq f(y)$ and $f(x) \sqsubseteq g(x)$, it follows that $f(x) \sqsubseteq f(y) \sqcap g(x)$. Thus, only the converse inequality has to be verified.

Because of our assumptions there are numbers i and j such that $f = [g]_i^{\rightarrow}$ and $x = [y]_j^K$, which means that we have to show that $[g(y)]_i^L \sqcap g([y]_j^K) \sqsubseteq [g([y]_j^K)]_i^L$.

Obviously $[g(y)]_i^L$, $g([y]_j^K) \sqsubseteq g(y)$. Hence, $[g(y)]_i^L$ and $g([y]_j^K)$ are comparable. Let us first assume that $[g(y)]_i^L \sqsubseteq g([y]_j^K)$. Then it follows that

$$[g(y)]_i^L \sqcap g([y]_j^K) = [g(y)]_i^L \sqsubseteq [g([y]_j^K)]_i^L.$$

Assume next that $g([y]_j^K) \sqsubseteq [g(y)]_i^L$. Then $g([y]_j^K) \in L_i$, which implies that $g([y]_j^K) = [g([y]_j^K)]_i^L$. Hence $[g(y)]_i^L \sqcap g([y]_j^K) = [g([y]_j^K)]_i^L$.

Note that the induced partial order on the rank-ordered set of all rank-preserving functions includes the pointwise ordering, but that the converse is not true, in general. An analogous remark holds with respect to the induced partial order on the product of two rank-ordered sets and the componentwise ordering. Nevertheless, least upper bounds in products and functions spaces, respectively, are defined component- and pointwise.

Next, for x, $y \in K$, set

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 2^{-\min\{\, i \mid [x]_i \neq [y]_i \,\}} & \text{otherwise,} \end{cases}$$

Proposition 4.5 *Let K be a rank-ordered set. Then (K,d) is a complete ultrametric space with $\bigcup\{\, K_i \mid i \in \omega \,\}$ as dense subspace.*

Proof: Obviously, $d(x,y) = d(y,x)$, and $d(x,y) = 0$ if and only if $x = y$. For the verification of the ultrametric inequation let x, y, $z \in K$ and without restriction assume that they are pairwise different. Then there exist $m = \max\{\, i \mid [x]_i = [y]_i \,\}$, $n = \max\{\, i \mid [y]_i = [z]_i \,\}$ and $r = \max\{\, i \mid [x]_i = [z]_i \,\}$. If $m \leq r$ then also $n \leq r$. Hence, $d(x,y) = 2^{-r-1} \leq \max\{2^{-m-1}, 2^{-n-1}\} = \max\{d(x,z), d(y,z)\}$. If $m > r$ then $n = r$. Thus, $d(x,y) = \max\{d(x,z), d(y,z)\}$. In the same way we obtain that $d(x,y) \leq \max\{d(x,z), d(y,z)\}$, if $n \leq r$ or $n > r$.

Next, let $(x_i)_{i\in\omega}$ be a Cauchy sequence in K. Then there are numbers n_i such that $d(x_m, x_n) < 2^{-i}$, for all m, $n \geq n_i$. Without restriction assume that all x_i are pairwise different. This implies that for all m, $n \geq n_i$, $\max\{\, j \mid [x_m]_j = [x_n]_j \,\} \geq i$. Let the n_i be minimal with this property. Then $n_{i+1} \geq n_i$, for all $i \in \omega$. Set $z_i = [x_{n_i}]_i$. It follows that

$$[z_{i+1}]_i = [[x_{n_{i+1}}]_{i+1}]_i = [x_{n_{i+1}}]_i = [x_{n_i}]_i = z_i.$$

Let z be the unique element of K with $[z]_i = z_i$, for all $i \in \omega$. Then we have for $n \geq n_i$ that $d(x_n, x_{n_i}) < 2^{-i}$ and $d(x_{n_i}, z) < 2^{-i}$. Thus

$$d(x_n, z) \leq \max\{d((x_n, x_{n_i}), d(x_{n_i}, z)\} < 2^{-i},$$

which implies that $z = \lim_n x_n$. This shows that (K, d) is complete.

Finally, for $x \in K$ and $i \in \omega$, let $B(x,i) = \{\, y \in K \mid d(x,y) < 2^i \,\}$. Then the collection of all $B(x,i)$ is a canonical basis of the metric topology on K. If $x = [x]_i$ then $x \in B(x,i) \cap K_i$. If $x \neq [x]_i$ then $\min\{\, j \mid [x]_j \neq [[x]_i]_j \,\} > i$, which implies that $d(x, [x]_i) < 2^{-i}$. This shows that $\bigcup\{\, K_i \mid i \in \omega \,\}$ is a dense base of K.

It follows that for finitely generated rank-ordered sets K, (K,d) is a complete separable ultrametric space.

As it is readily verified, the rank-preserving functions between rank-ordered sets are just the nonexpansive functions between them. Thus, they are continuous with respect to the metric topology.

Propositions 4.5 and 4.3 seem to be folklore [8]. Their proofs have been included for completeness reasons only. From these propositions we obtain that on a rank-ordered set there are at least two canonical topologies: the metric topology and the Scott topology. As it is well known, on Scott domains D there is another canonical topology, the *Lawson topology*, which is generated by the subsets of D which are either Scott open or of the form $D\backslash\uparrow\{x\}$, for some $x \in D$, where $\uparrow\{x\} = \{ y \in D \mid x \sqsubseteq y \}$.

Proposition 4.6 *The metric topology on a rank-ordered set is equivalent to the Lawson topology and hence finer than the Scott topology.*

Proof:

Claim 1 The metric topology is coarser than the Lawson topology.

Let $z \in K^0$ and $n \in \omega$. Since for $x \in B(z,n)$ we have that $[z]_n^K = [x]_n^K$ and hence that $[z]_n^K \sqsubseteq x$, it follows that $B(z,n) \subseteq O_{[z]_n^K}$. If both sets are equal, we are done. In the other case we show that

$$B(z,n) = O_{[z]_n^K} \cap \bigcup \{ K\backslash\uparrow\{y\} \mid [z]^K_{\mathrm{rk}([z]_n^K)} \sqsubset y \wedge [y]_n^K \neq [z]_n^K \}.$$

Here, for u, $v \in K$, $u \sqsubset v$ means that $u \sqsubseteq v$ and $u \neq v$.

Since $B(z,n) \neq O_{[z]_n^K}$ there is some $y \in K$ such that $[y]_n^K \neq [z]_n^K$ and $[z]^K_{\mathrm{rk}([z]_n^K)} \sqsubset y$. Now, let $x \in B(z,n)$ and assume that $y \sqsubseteq x$.

Obviously $x \neq y$. As $y = [x]^K_{\mathrm{rk}(y)}$ and $[x]_n^K = [z]_n^K$, it follows that $[x]_n^K \neq [y]_n^K = [x]^K_{\min\{n,\mathrm{rk}(y)\}}$, which implies that $\mathrm{rk}(y) < n$. Moreover, because $[z]^K_{\mathrm{rk}([z]_n^K)} \sqsubset y$ we obtain that $[y]^K_{\mathrm{rk}([z]_n^K)} = [z]^K_{\mathrm{rk}([z]_n^K)}$ and $\mathrm{rk}([z]_n^K) < \mathrm{rk}(y)$. Thus, we have

$$[z]_n^K = [z]^K_{\mathrm{rk}([z]_n^K)} \sqsubseteq [z]^K_{\mathrm{rk}(y)} \sqsubseteq [z]_n^K.$$

This shows that $[z]_n^K = [z]^K_{rk(y)}$. Since $\mathrm{rk}(y) < n$ it follows that $[y]_n^K = y = [x]^K_{\mathrm{rk}(y)} = [z]^K_{\mathrm{rk}(y)} = [z]_n^K$, in contradiction to the above mentioned properties of y. Thus $y \not\sqsubseteq x$, which means that $x \in K\backslash\uparrow\{y\}$.

Now, for the verification of the converse inclusion, let $x \in O_{[z]_n^K} \cap \bigcup\{ K\backslash\uparrow\{y\} \mid [z]^K_{\mathrm{rk}([z]_n^K)} \sqsubset y \wedge [y]_n^K \neq [z]_n^K \}$. We have to show that $[x]_n^K = [z]_n^K$. Obviously $[z]_n^K \sqsubseteq x$. If $x = [z]_n^K$ then $[x]_n^K = [z]_n^K$ and we are done. Assume that $x \neq [z]_n^K$ and note that $[z]_n^K = [z]^K_{\mathrm{rk}([z]_n^K)}$. Thus $[z]^K_{\mathrm{rk}([z]_n^K)} \sqsubset x$, which implies that $[x]_n^K = [z]_n^K$.

Claim 2 The Lawson topology is coarser than the metric topology.

We first show that the Scott topology is coarser than the metric topology. Let $z \in K_n$ and let n be minimal with this property. Then we have for $x \in K$ that

$$z \sqsubseteq x \quad \Leftrightarrow \quad z = x \vee z = [x]_n \quad \Leftrightarrow \quad [z]_n = [x]_n \quad \Leftrightarrow \quad d(z,x) < 2^{-n},$$

which shows that $O_z = B(z,n)$.

It remains to prove that also each set $K\backslash\uparrow\{y\}$ $(y \in K)$ is open in the metric topology. Let $y \neq \perp_K$ and for each $z \in K$ which is different from y, set $\mu(y,z) = \min\{ i \mid [y]_i^K \neq [z]_i^K \}$. Then we show that

$$K\backslash\uparrow\{y\} = \bigcup \{ B(z, \mu(y,z)) \mid z \in K^0 \wedge y \not\sqsubseteq z \}.$$

Let $x \in K$ such that $y \not\sqsubseteq x$. It follows that $y \neq x$. Let $\bar{n} = \mu(y,x)$ and set $z = [x]^K_{\bar{n}}$. Then $\mu(y,z) \leq \bar{n} \leq \mathrm{rk}(y)$.

If $y \sqsubseteq z$ it would follow that $y = [z]^K_{\mathrm{rk}(y)} = [[x]^K_{\bar{n}}]^K_{\mathrm{rk}(y)} = [x]^K_{\bar{n}}$ and hence that $[y]^K_{\bar{n}} = [x]^K_{\bar{n}}$, in contradiction to the definition of $\bar{n}$.

Thus $y \not\sqsubseteq z$. Moreover, since $[x]^K_{\bar{n}} = [z]^K_{\bar{n}}$, we have that also $[x]^K_{\mu(y,z)} = [z]^K_{\mu(y,z)}$, which shows that $x \in B(z, \mu(y,z))$.

For the proof of the converse inclusion let $x \in B(z, \mu(y,z))$, for some $z \in K^0$ with $y \not\sqsubseteq z$. Assume that $y \sqsubseteq x$. Then $y = x$ and hence $y \in B(z, \mu(y,z))$, which is impossible by the definition of $\mu(y,z)$, or $y = [x]^K_{\mathrm{rk}(y)}$ and thus $[y]^K_{\bar{n}} = [x]^K_{\bar{n}}$, which is impossible as well. Hence $x \in K \backslash \uparrow\{y\}$.

As we have already seen, rank-preserving functions are continuous with respect to the metric topology. The next example shows that they are not continuous with respect to the Scott topology: They are not monotone. But as we shall see then, it is just this property which is missing for their being Scott continuous.

Example 4.7 Let $K = \{\bot, A, \top\}$ with $[\cdot]^K_i = id_K$, for $i \geq 2$, and

$$[\top]^K_1 = [A]^K_1 = A, \quad [\bot]^K_1 = \bot,$$
$$[\top]^K_0 = [A]^K_0 = [\bot]^K_0 = \bot.$$

Moreover, let $L = \{\bot, a, b, c\}$ with $[\cdot]^L_i = id_L$, for $i \geq 2$, and

$$[c]^L_1 = [b]^L_1 = [a]^L_1 = a, \quad [\bot]^L_1 = \bot,$$
$$[c]^L_0 = [b]^L_0 = [a]^L_0 = [\bot]^L_0 = \bot.$$

Define $f\colon K \to L$ by

$$f(\bot) = \bot, \ f(A) = b, \ f(\top) = c.$$

Then f is rank-preserving. However, f is not monotone and hence not Scott continuous: We have that $A \sqsubseteq_K \top$, $f(A) = b$, and $f(\top) = c$. But $b \not\sqsubseteq_L c$.

Lemma 4.8 *A rank-preserving function is Scott continuous if and only if it is monotone.*

Proof: We only have to show the "if"-part. By Proposition 4.6 every rank-preserving function is Lawson continuous. Hence, the inverse image of a Scott open set is Lawson open. If, in addition, the function is monotone, it is also upwards closed with respect to the partial order. Because of the special form of the Lawson open sets it follows that in this case the inverse image of a Scott open set is Scott open again.

5 Rank-Ordered Sets and Limits

In Section 1 we have seen that each rank-ordered set K is isomorphic to the inverse limit $\vec{K}$ of the cochain $(K_i, p_i)_{i \in \omega}$, with $p_i = [\cdot]^K_i \restriction K_{i+1}$. Let $f \in [K \Rightarrow \vec{K}]$ be the isomorphism. As it is well known, on each inverse limit space there is a canonical topology induced by the topology of the component spaces. Let $\vec{T}$ be the inverse limit of the cochain $(T_i, q_i)_{i \in \omega}$ of topological spaces, let $q_{\infty i}$ be the canonical projection of $\vec{T}$ onto T_i, and let $\mathcal{B}_i$ be a basis for the topology on T_i. Then $\vec{\mathcal{B}} = \{\, q^{-1}_{\infty i}(B) \mid B \in \mathcal{B}_i \wedge i \in \omega \,\}$ is a basis for the canonical topology on $\vec{T}$.

Proposition 5.1 *Let K be a rank-ordered set and let the K_i be furnished with the discrete topology. Then the following hold:*

1. *If K is supplied with its canonical metric topology and $\vec{K}$ with the canonical topology induced by the topology on the K_i's, then f is a homeomorphism.*

2. *If K is supplied with its canonical Scott topology and $\vec{K}$ with the topology generated by the sets $p_{\infty 0}^{-1}(\perp_K)$ and $p_{\infty i}^{-1}(\{a\})$, for $a \in K_{i+1} \setminus K_i$ and $i \in \omega$, then f is a homeomorphism.*

Proof: (1) Let $a \in K_i$, for $i \in \omega$. Then

$$f^{-1}(p_{\infty i}^{-1}(\{a\})) = \{\, x \in K \mid [x]_i^K = a \,\} = \{\, x \in K \mid [x]_i^K = [a]_i^K \,\} = B(a, i).$$

Next, let $x \in K$ and $i \in \omega$. Then we have for the inverse mapping $g \in [\vec{K} \Rightarrow K]$ of f that

$$\begin{aligned} g^{-1}(B(x,i)) &= g^{-1}(\{\, y \in K \mid (\forall j \leq i)[x]_j^K = [y]_j^K \,\} \\ &= \{\, ([y]_j^K)_{j\in\omega} \in \vec{K} \mid [x]_i^K = [y]_i^K \,\} \;=\; p_{\infty i}^{-1}(\{[x]_i^K\}). \end{aligned}$$

It follows that both f and f^{-1} are continuous with respect to the metric topology on K and the canonical topology on $\vec{K}$.

(2) Let $a \in K_{i+1} \setminus K_i$, for some $i \geq 1$. As we have seen in the proof of Proposition 4.6, $O_a = B(a, i)$ in this case. Hence, it follows from (1) that $f^{-1}(p_{\infty i}^{-1}(\{a\})) = O_a$. If $a \in K_0$, then $p_{\infty i}^{-1}(\{a\}) = \vec{K}$, *i. e.*, $f^{-1}(p_{\infty i}^{-1}(\{a\})) = K$.

Now, let $x \in K_i$, for some $i \in \omega$, and let i be minimal with this property. Then we have that $g^{-1}(O_x) = p_{\infty i}^{-1}(\{[x]_i^K\}) = p_{\infty i}^{-1}(\{x\})$.

Again it follows that f and f^{-1} are continuous, this time with respect to the Scott topology on K and the topology on $\vec{K}$ generated by the sets $p_{\infty i}^{-1}(\{a\})$, for $i \geq 0$ and $a \in K_i$ such that $a \notin K_{i-1}$, if $i > 0$.

Definition 5.2 Let $(M^i, p_i)_{i\in\omega}$ and $((M')^i, p_i')_{i\in\omega}$ be two cochains of sets with inverse limits M and M', respectively. A sequence $(f_i)_{i\in\omega}$ with $f_i\colon M^i \to (M')^i$ is a *sequent morphism* from M to M' if for all $i \in \omega$

$$f_i \circ p_i = p_i' \circ f_{i+1}.$$

Application and composition of sequent morphisms are defined componentwise. Note that for $x \in M$, $p_i'(f_{i+1}(x(i+1))) = f_i(p_i(x(i+1))) = f_i(x(i))$, which means that $(f_i(x(i)))_{i\in\omega} \in M'$.

Let **IL** be the category which has

- as objects inverse limits of cochains $(M^i, p_i)_{i\in\omega}$ such that M^0 is a singleton set and for every $i \in \omega$ there is a map $e_i\colon M^i \to M^{i+1}$ with $p_i \circ e_i = \mathrm{id}_{M^i}$ and

- as morphisms the corresponding sequent morphisms.

The product of two objects is the inverse limit of the cochain which is obtained from the given cochains by taking the product componentwise and the terminal object is the inverse limit of the cochain all components of which are equal to the same singleton set. In what follows we will show that **RS** is equivalent to **IL**. But, first, we prove that **IL** is Cartesian closed.

For two cochains $(M^i, p_i)_{i\in\omega}$ and $((M')^i, p_i')_{i\in\omega}$ let

$$\begin{aligned} &[M^i \to (M')^i] = \\ &\quad \{\, g\colon M^i \to (M')^i \mid (\forall j < i)(\forall x, y \in M^i)[p_{ij}(x) = p_{ij}(y) \Rightarrow p_{ij}'(g(x)) = p_{ij}'(g(y))] \,\}. \end{aligned}$$

Moreover, for $g \in [M^{i+1} \to (M')^{i+1}]$ and $f \in [M^i \to (M')^i]$, let $q_i(g) = p_i' \circ g \circ e_i$ and $d_i(f) = e_i' \circ f \circ p_i$. Then $q_i(g) \in [M^i \to (M')^i]$, $d_i(f) \in [M^{i+1} \to (M')^{i+1}]$, and $q_i(d_i(f)) = f$. It follows that the inverse limit $[M \to M']$ of $([M^i \to (M')^i], q_i)_{i\in\omega}$ is in object in **IL**.

For $f \in [M^i \to (M')^i]$ and $y \in M^i$ set

$$\mathrm{ev}_i(f, y) = f(y).$$

Then $\mathrm{ev}_i \circ (q_i \times p_i) = p'_i \circ \mathrm{ev}_{i+1}$. Hence, $(\mathrm{ev}_i)_{i\in\omega}$ is a sequent morphism.

Now, let $(f_i)_{i\in\omega}$ be a sequent morphism from $M'' \times M$ to M' and for $x \in (M'')^i$ let $\hat{f}_i(x)$ be the function that maps $y \in M^i$ to $f_i(x, y)$. Then $\hat{f}_i(x) \in [M^i \to (M')^i]$ and $(\hat{f}_i)_{i\in\omega}$ is a sequent morphism from $((M'')^i, p''_i)_{i\in\omega}$ to $([M^i \to (M')^i], q_i)_{i\in\omega}$. Moreover, it is the uniquely determined morphism $(h_i)_{i\in\omega}$ such that $\mathrm{ev}_i \circ (h_i \times \mathrm{id}_{M^i}) = f_i$, for $i \in \omega$. This shows that $[M \to M']$ is an exponent of M and M' in **IL**.

Theorem 5.3 **IL** *is Cartesian closed.*

As we have seen in Theorem 1.3 every object in **IL** is rank-ordered. Moreover, every morphism $\vec{f} = (f_i)_{i\in\omega}$ in **IL** is rank-preserving. In order to see this, let $x \in M$ and $m, n \in \omega$ with $n \geq m$. Then we have

$$[\vec{f}([x]_n^M)]_m^{M'} = \mathrm{in}'_m(\vec{f}(\mathrm{in}_n(x(n)))(m)) = \mathrm{in}'_m(f_m(x(m))) = \mathrm{in}'_m(\vec{f}(x)(m)) = [\vec{f}(x)]_m^{M'}.$$

Proposition 5.4 **IL** *is a full subcategory of* **RS**.

Proof: In order to see that the inclusion functor $I\colon \mathbf{IL} \to \mathbf{RS}$ is full, let M and M' be objects in **IL** and $h \in [M \Rightarrow M']$. For $y \in M^i$ set $f_i(y) = h(\mathrm{in}_i(y))(i)$.

First, we verify that $\vec{f} = (f_i)_{i\in\omega}$ is a sequent morphism. Let $z \in M^{i+1}$. Then

$$\begin{aligned} p'_i(h(\mathrm{in}_{i+1}(z))(i+1)) &= h(\mathrm{in}_{i+1}(z))(i) = \mathrm{in}'_i(h(\mathrm{in}_{i+1}(z))(i))(i) = [h(\mathrm{in}_{i+1}(z))]_i^{M'}(i) \\ &= [h([\mathrm{in}_{i+1}(z)]_i^M)]_i^{M'}(i) = [h(\mathrm{in}_i(p_i(z)))]_i^{M'}(i) = h(\mathrm{in}_i(p_i(z)))(i). \end{aligned}$$

Next, we show that $\vec{f} = h$. Let $x \in M$. Then we have

$$\vec{f}(x)(i) = f_i(x(i)) = h(\mathrm{in}_i(x(i)))(i) = h([x]_i^M)(i) = [h([x]_i^M)]_i^{M'}(i) = [h(x)]_i^{M'}(i) = h(x)(i).$$

Since every rank-ordered set K is isomorphic (in **RS**) to an object in **IL**, namely $\vec{K}$, we have the ensuing consequence [10].

Theorem 5.5 *1. The categories* **IL** *and* **RS** *are equivalent.*

2. **IL** *is reflective in* **RS**.

It follows that the inclusion functor commutes with the product and the exponent functor, *i. e.*,

$$I(M \times_{IL} M') \cong I(M) \times_{RS} I(M') \quad \text{and} \quad I([M \to M']) \cong [I(M) \Rightarrow I(M')].$$

In the special case that M and M' are representations of rank-ordered sets K and L we obtain that $[K \Rightarrow L] \cong I([\vec{K} \to \vec{L}])$ and hence that $[K \Rightarrow L]_i \cong [K_i \to L_i]$. Note that in this case $[K_i \to L_i] = \{ g\colon K_i \to L_i \mid (\forall j \leq i)(\forall x \in K_i)[g(x)]_j^L = [g([x]_j^K)]_j^L \}$. Moreover, for $g \in [K_{i+1} \to L_{i+1}]$, $f \in [K_i \to L_i]$, $x \in K_i$, and $y \in K_{i+1}$, $q_i(g)(x) = [g(x)]_i^L$ and $d_i(f)(y) = f([y]_i^K)$.

Lemma 5.6 *Let K and L be rank-ordered sets. Then the following statements hold:*

1. *The sets $[K \Rightarrow L]_i$ and $[K_i \to L_i]$ are isomorphic.*
2. *$[K \Rightarrow L]$ is isomorphic to the inverse limit of the cochain $([K_i \to L_i], q_i)_{i\in\omega}$.*

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