Effectively Given Spaces, Domains, and Formal Topology

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In this talk we study the relationship between effectively given spaces and domains on the one side and formal topology on the other.

1. Effectively given spaces as formal spaces.
2. (Continuous) domains and formal topology.

1. Effectively given spaces as formal spaces

Definition

A *formal topology* $\mathcal{S} = (S, \leq, \lhd, \mathrm{Pos})$ *with preorder*, short: a $\leq$-*formal topology*, consists of

- a set S,
- a preorder $\leq$ over S,
- a relation $\lhd$, called *cover*, between elements and subsets of S, and
- a predicate Pos, called *positivity*, over S such that the following conditions hold:

(reflexivity) $$\frac{a \in U}{a \lhd U}$$ (transitivity) $$\frac{a \lhd U \quad U \lhd V}{a \lhd V}$$

($\leq$-left) $$\frac{a \leq b \quad b \lhd U}{a \lhd U}$$ ($\leq$-right) $$\frac{a \lhd U \quad a \lhd V}{a \lhd {\downarrow}U \cap {\downarrow}V},$$

where

$$U \lhd V \Leftrightarrow (\forall u \in U) u \lhd V$$

$$\downarrow U = \{ c \in S \mid (\exists u \in U) c \leq u \}$$

(monotonicity) $$\frac{\mathrm{Pos}(a) \quad a \lhd U}{(\exists u \in U)\mathrm{Pos}(u)}$$ (positivity) $$\frac{\mathrm{Pos}(a) \to a \lhd U}{a \lhd U}$$

Definition

A subset α of S is a *formal point* of $\mathcal{S}$ if the following requirements are satisfied:

(inhabited) $$(\exists s \in S) s \in \alpha$$

(point convergence) $$\frac{a \in \alpha \quad b \in \alpha}{(\exists c \in \alpha) c \leq a \& c \leq b}$$

(point splitness) $$\frac{a \in \alpha \quad a \lhd U}{(\exists u \in U) u \in \alpha}$$

(point positivity) $$\frac{a \in \alpha}{\mathrm{Pos}(a)}$$

A cover relation $\lhd$ can be inductively generated if we are given

- a preordered set $(S, \leq)$
- a family $I(a)$ $(a \in S)$ of sets
- a family $C(a, i) \subseteq S$ $(a \in S, i \in I(a))$ such that
  - for any $a \leq c$ and $i \in I(c)$ there is some $j \in I(a)$ such that

$$C(a, j) \subseteq {\downarrow} a \cap {\downarrow} C(c, i)$$

    (i.e, (I, C) is *located*)
- a predicate Pos on S satisfying
  - $\mathrm{Pos}(a) \& a \leq b \rightarrow \mathrm{Pos}(b)$
  - $\mathrm{Pos}(a) \& i \in I(a) \rightarrow \mathrm{Pos}(C(a, i))$.

In this case a cover relation $\lhd$ can be inductively generated by the rules *reflexivity*, $\leq$-*left*, *positivity* and

$$\text{(infinity)} \quad \frac{i \in I(a) \quad C(a,i) \lhd U}{a \lhd U}$$

(substituting *transitivity*) so that $(S, \leq, \lhd, \mathrm{Pos})$ is a formal topology and $\lhd$ is the least cover $\lhd'$ with

$$a \lhd' C(a,i) \; (a \in S, i \in I(a))$$

making $(S, \leq, \lhd', \mathrm{Pos})$ into a formal topology.

Apply this technique to generate a cover in the case of effectively given spaces.

Definition

Let $\mathcal{T} = (T, \tau)$ be a topological T_0 space with countable base $\mathcal{B}$ of the topology. Let $B \colon \omega \to \mathcal{B}$ (onto) be an indexing of $\mathcal{B}$ and $\prec \subseteq \omega \times \omega$ be a transitive relation (on names) (called *strong inclusion*) such that

- $m \prec n \Rightarrow B_m \subseteq B_n$
- $(\forall z \in T)(\forall a, b \in \omega)[z \in B_a \& z \in B_b \Rightarrow (\exists c \in \omega) z \in B_c \& c \prec a \& c \prec b]$.

For $z \in T$, let $\mathcal{N}(z) = \{ c \in \omega \mid z \in B_c \}$.

$\mathcal{T}$ is called *effective* if

- $\prec$ is computably enumerable
- $\mathcal{N}(z)$ is computably enumerable, for all $z \in T$.

Some similarities between effective spaces and formal topology are already clear from this definition. In both cases computations and/or mathematical reasoning is done on the side of basic open sets, not on points. Moreover, it is actually not the basic open sets itself which are in the centre of interest, but their names (codes). So, effective spaces will not be considered as formal topologies in the way concrete spaces are in general.

- Set $S = \omega$ and

$$a \preceq b \Leftrightarrow a \prec b \text{ or } a = b.$$

- For $a \in S$, let $I(a) = \{*\}$ and

$$C(a, *) = {\downarrow_{\prec}} a = \{\, c \in S \mid c \prec a \,\}.$$

Then (I, C) is localized:

Let $a \preceq c$. Then $C(a, *) = {\downarrow_{\prec}} a$ and

$$ {\downarrow_{\preceq}} a \cap {\downarrow_{\preceq}} C(c, *) = (\{a\} \cup {\downarrow_{\prec}} a) \cap {\downarrow_{\prec}} c = \begin{cases} {\downarrow_{\prec}} a & \text{if } a = c, \\ {\downarrow_{\preceq}} a & \text{if } a \prec c. \end{cases} $$

Thus $C(a, *) \subseteq {\downarrow_{\preceq}} a \cap {\downarrow_{\preceq}} C(c, *)$.

- Let
$$\mathrm{Pos}(a) \Leftrightarrow (\exists z \in T) z \in B_a.$$
Then
  1. $\mathrm{Pos}(a) \& a \preceq b \Rightarrow \mathrm{Pos}(b)$
  2. $\mathrm{Pos}(a) \& i \in I(a) \Rightarrow (\exists b \in C(a,i)) \mathrm{Pos}(b)$.

For (1): Remember that $a \preceq b \Rightarrow B_a \subseteq B_b$.
For (2): Since $B_a = \bigcup \{ B_c \mid c \prec a \}$.

- Let $\lhd$ be inductively generated by the axiom set (I, C) and the above mentioned rules.

Then $\mathcal{F}(\mathcal{T}) = (S, \preceq, \lhd, \mathrm{Pos})$ is a formal topology.

Proposition

For all $z \in T$, $\mathcal{N}(z)$ is a computably enumerable formal point of $\mathcal{F}(\mathcal{T})$.

In general, there will be more formal points with respect to $\mathcal{F}(\mathcal{T})$ than there are points in $\mathcal{T}$. Let us restrict ourselves to those formal points that are computably enumerable. Then we have a nice property.

Lemma

Let α be a computably enumerable formal point of $\mathcal{F}(\mathcal{T})$. Then there is a computable sequence $(a_i)_{i\in\omega}$ descending with respect to $\prec$ such that

$$\alpha = \{\, b \in S \mid (\exists i \in \omega) a_i \prec b \,\}.$$

Definition

An effective space $\mathcal{T}$ is *constructively complete*, if for every $\prec$-descending computable sequence $(a_i)_{i \in \omega}$ $(\subseteq \omega)$ there is some $z \in T$ such that

$$\mathcal{N}(z) = \{ b \in S \mid (\exists i \in \omega) a_i \prec b \}.$$

Proposition

The computably enumerable formal points of $\mathcal{F}(\mathcal{T})$ are exactly the sets $\mathcal{N}(z)$ with $z \in T$ if, and only if, $\mathcal{T}$ is constructively complete.

Definition

Let $(D, \sqsubseteq, \bot)$ be a poset with least element $\bot$.

1. D is *directed-complete* if every directed subset S of D has a least upper bound $\bigsqcup S$ in D.
2. For elements $x, y \in D$, x *approximates* y ($x \ll y$), if for every directed subset S of D,

$$y \sqsubseteq \bigsqcup S \Rightarrow (\exists s \in S) x \sqsubseteq s.$$

3. An element x of D is *compact* if $x \ll x$. Let K^D denote the set of compact elements of D.
4. D is an *algebraic domain*, if
   - D is directed-complete and
   - for every $x \in D$, $K^D_x = \{ z \in K^D \mid z \sqsubseteq x \}$ is directed and

$$x = \bigsqcup K^D_x.$$

Proposition

Up to isomorphism, every algebraic domain is the set of ideals over the poset $(K^D, \sqsubseteq, \bot)$ *ordered by set inclusion.*

In this case we will use formal topologies where the preorder has been chosen in a particular way.

Definition

A *quasi formal topology* $\mathcal{S} = (S, \top, \lhd, \mathrm{Pos})$ over a set S consists of

- an element $\top$ of S
- a relation $\lhd$ between elements and subsets of S and
- a predicate Pos over S such that the following conditions hold:

(reflexivity) $$\frac{a \in U}{a \lhd U}$$ (transitivity) $$\frac{a \lhd U \quad U \lhd V}{a \lhd V}$$

(monotonicity) $$\frac{\mathrm{Pos}(a) \quad a \lhd U}{(\exists u \in U)\mathrm{Pos}(u)}$$ (positivity) $$\frac{\mathrm{Pos}(a) \to a \lhd U}{a \lhd U}$$

$$a \lhd \{\top\}$$

Definition

A quasi formal topology $\mathcal{S}$ is *unitary* if

$$a \lhd U \leftrightarrow [\mathrm{Pos}(a) \rightarrow (\exists b \in U) a \lhd \{b\}].$$

Definition

A *formal topology* $\mathcal{S}$ is a quasi formal topology for which in addition

$$(\downarrow\text{-right}) \quad \frac{a \lhd U \quad a \lhd V}{a \lhd U{\downarrow}V}$$

is satisfied, where

$$U{\downarrow}V = \{\, d \in S \mid (\exists u \in U)(\exists v \in V) d \lhd \{u\} \& d \lhd \{v\} \,\}.$$

Proposition

Let $\mathcal{S} = (S, \top, \lhd, \mathrm{Pos})$ *be a quasi formal topology and define*

$$a \leq b \Leftrightarrow a \lhd \{b\}.$$

Then $(S, \leq, \top)$ *is a preordered set with greatest element and hence* $(S, \leq^{op}, \top)$ *is a preordered set with least element.*

If, in addition, $\mathcal{S}$ *is unitary, then* $\mathcal{S}$ *is a formal topology the formal points of which are exactly the filters over* $(S, \leq)$*, i.e. the ideals over* $(S, \leq^{op})$*.*

Corollary

The formal points of a unitary quasi formal topology form an algebraic domain with respect to set inclusion.

Conversely:

Proposition

Let $(S, \leq, \top)$ *be a preordered set with greatest element and define*

$$a \lhd U \Leftrightarrow (\exists u \in U) a \leq u.$$

Then $\mathcal{S} = (S, \top, \lhd, \mathbf{true})$ *is a unitary formal topology the formal points of which are exactly the filters over* $(S, \leq)$.

This is all well-known. Let us now come to the case of continuous domains.

Definition

Let $(D, \sqsubseteq, \bot)$ be a directed-complete partial order with least element.

1. A subset B of D is a *basis*, if for every $x \in D$ the set $B_x = \{ z \in B \mid z \ll x \}$ is directed and $x = \bigsqcup B_x$.
2. D is a *continuous domain* if it has a basis.

Proposition

1. *Let B be a basis of some continuous domain. Then $(B, \ll)$ is an* abstract basis, *i.e.*
   - *$\ll$ is transitive*
   - $x \ll z \& y \ll z \Rightarrow (\exists u \in B) x, y \ll u \ll z$ *(interpolation prop.)*
2. *Up to isomorphism, every continuous domain is the set of ideals over some abstract basis ordered by set inclusion.*

Definition

1. A *weak formal topology* $\mathcal{S} = (S, \top, \lhd, \mathrm{Pos})$ over a set S consists of
   - an element $\top$ of S
   - a relation $\lhd$ between elements and subsets of S and
   - a predicate Pos over S

   such that transitivity, $\downarrow$-right, monotonicity and positivity holds.
2. A weak formal topology $\mathcal{S}$ is *unitary* if

$$a \lhd U \leftrightarrow [\mathrm{Pos}(a) \rightarrow (\exists b \in U) a \lhd \{b\}].$$

Note that *reflexivity* is no longer required to hold!

Proposition

Let $\mathcal{S} = (S, \top, \lhd, \mathrm{Pos})$ *be a unitary weak formal topology and set*

$$a \prec b \Leftrightarrow a \lhd \{b\}.$$

Then $(S, \prec^{op})$ *is an abstract basis and the ideals over* $(S, \prec^{op})$ *are exactly the formal points of* $\mathcal{S}$*.*

Corollary

The formal points of a unitary weak formal topology form a continuous domain with respect to set inclusion.

Conversely:

Proposition

Let $(S, \prec, \bot)$ *be an abstract basis with least element and define*

$$a \lhd U \Leftrightarrow (\exists u \in U) a \prec^{op} u.$$

Then $\mathcal{S} = (S, \bot, \lhd, \mathbf{true})$ *is a unitary weak formal topology the formal points of which are exactly the ideals of* $(S, \prec)$.

Let us now consider the morphisms between weak formal topologies and/or abstract bases.

Definition

Let $\mathcal{S} = (S, \top_S, \lhd_S, \mathrm{Pos}_S)$ and $\mathcal{T} = (T, \top_T, \lhd_T, \mathrm{Pos}_T)$ be weak formal topologies. A binary relation F between S and T is *continuous* if the following conditions hold:

(function totality) $\qquad (\forall x \in S)(\exists y \in T)xFy$

(function convergence) $$\frac{aFb \quad aFd}{a \lhd_S F^{-}(\{b\}{\downarrow}\{d\})}$$

(function saturation) $$\frac{a \lhd_S W \quad WFb}{aFb}$$

(function continuity) $$\frac{aFb \quad b \lhd_T V}{a \lhd_S F^{-}(V)}.$$

Here, $F^{-}(V) = \{\, c \in S \mid (\exists v \in V)cFv \,\}$.

Definition

Let $(B, \prec_B, \perp_B)$ and $(C \prec_C, \perp_C)$ be abstract bases with least elements. A binary relation R between B and C is *approximable* if the following conditions hold:

1. $\perp_B R \perp_C$
2. $\dfrac{xRy \quad y' \prec_C y}{xRy'}$
3. $\dfrac{xRy \quad xRy'}{(\exists z \in C) xRz \& y, y' \prec_C z}$
4. $\dfrac{x \prec_B x' \quad xRy}{x'Ry}$
5. $\dfrac{xRy}{(\exists z \in C) z \prec_B x \& zRy}$.

Proposition

Let $\mathcal{S} = (S, \top_S, \lhd_S, \mathrm{Pos}_S)$ *and* $\mathcal{T} = (T, \top_T, \lhd_T, \mathrm{Pos}_T)$ *be unitary weak formal topologies. Define*

$$a \prec_S b \Leftrightarrow a \lhd_S \{b\} \quad \text{and} \quad c \prec_T d \Leftrightarrow c \lhd_T \{d\}.$$

Moreover, let F *be a relation between* S *and* T. *Then* F *is continuous with respect to* $\mathcal{S}$ *and* $\mathcal{T}$ *if, and only if,* F *is approximable with respect to* $(S, \prec_S^{op})$ *and* $(T, \prec_T^{op})$.

Proposition

Let $(B, \prec_B, \bot_B)$ *and* $(C, \prec_C, \bot_C)$ *be abstract bases with least element and define*

$$a \lhd_B U \Leftrightarrow (\exists u \in U) a \prec_B^{op} u. \quad \text{and} \quad c \lhd_C V \Leftrightarrow (\exists v \in V) c \prec_C^{op} v.$$

Moreover, let R *be a relation between* B *and* C*. Then* R *is approximable with respect to* $(B, \prec_B, \bot_B)$ *and* $(C, \prec_C, \bot_C)$ *if, and only if,* R *is continuous with respect to* $(B, \bot_B, \lhd_B, \mathbf{true})$ *and* $(C, \bot_C, \lhd_C, \mathbf{true})$*.*