Solving Domain Equations in the Category of Effectively Given Topological Spaces

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1. Introduction

The possibility to solve recursive domain equations is an important requirement for categories of spaces used in model constructions for higher-order logic and programming language semantics.

Here, we show how the techniques that are used to solve such equations in the categories of complete partial orders can be modified in order to obtain such solutions in the category of effectively given topological spaces.

2. Framework

Let $\mathcal{T} = (T, \tau)$ be a topological T_0 space with countable basis

$$\mathcal{B} = \{ B_0, B_1, \dots \}$$

and

$$\prec_B \subseteq \omega \times \omega$$

be transitive.

Definition

1. $\prec_B$ is a *strong inclusion* if
$$m \prec_B n \Rightarrow B_m \subseteq B_n.$$
2. The space $\mathcal{T} = (T, \tau, \mathcal{B}, \prec_B)$ is *effectively given* if the strong inclusion relation $\prec_B$ is semi-decidable.

Note that $\prec_B$ is not semi-decidable, in general, if we choose $\prec_B$ to be set inclusion.

3. Standard Examples

3.1 Effectively given metric spaces.

Let $\mathcal{M} = (M, \delta)$ be a separable metric space with dense subspace

$$M_0 = \{ x_0, x_1, \ldots \}.$$

Set

$$B_{\langle i,m \rangle} = \{ y \in M \mid \delta(x_i, y) < 2^{-m} \}$$

and

$$\langle i, m \rangle \prec_B \langle j, n \rangle \Leftrightarrow \delta(x_i, x_j) + 2^{-m} < 2^{-n}.$$

Note that the relation $<$ on the computable real numbers is semi-decidable.

Definition

$\mathcal{M}$ is *effectively given* if

- $\delta \colon M_0 \times M_0 \to \mathbb{R}_{\mathrm{comp}}$,
- There is some computable function $d \in R^{(2)}$ so that

$$\delta(x_i, x_j) = \gamma_{d(i,j)},$$

where γ is a standard indexing of the computable reals.

3.2 Effectively given domains

Let $\mathcal{D} = (D, \sqsubseteq)$ be an ω-continuous complete partial order (domain) with basis

$$D_0 = \{ z_0, z_1, \ldots \}.$$

Set

$$B_m = \{ y \in D \mid z_i \ll y \}$$

and

$$m \prec_B n \Leftrightarrow z_n \ll z_m.$$

Here, $\ll$ is the way-below relation.

Definition

$\mathcal{D}$ is *effectively given* if the way-below relation $\ll$ is semi-decidable.

4. The Smyth-Plotkin approach

Let $\mathbb{C}$ be a category and $F\colon \mathbb{C} \to \mathbb{C}$ be a functor.

The equation

$$X = F(X)$$

has a solution in $\mathbb{C}$ if

- $\mathbb{C}$ has an initial object $\bot$.
- $\bot \underset{e_{\bot,F(\bot)}}{\longrightarrow} F(\bot) \underset{F(e_{\bot,F(\bot)})}{\longrightarrow} F^2(\bot) \longrightarrow \cdots$ has a colimit.
- F is ω-continuous.

Consider the chain

$$T_0 \underset{e_0}{\longrightarrow} T_1 \underset{e_1}{\longrightarrow} T_2 \longrightarrow \cdots ,$$

where

- $\mathcal{T}_a$ effectively given T_0 space,
- e_a effectively continuous and 1-1.

Goal: Construction of a colimit.

Construction of a colimit

- Take
$$\bigoplus_{a\in\omega} T_a \qquad \text{(disjoint union)}$$
with the canonical topology.
- Set
$$T^\infty = \bigoplus_{a\in\omega} T_a / \sim$$
with the quotient topology,

where

$$(a, y) \sim (c, z) \Leftrightarrow [a \leq c \wedge e_{ac}(y) = z] \vee [c < a \wedge e_{ca}(z) = y]$$

and

$$e_{ac} = e_{c-1} \circ \cdots \circ e_a, \text{ for } a < c, \quad e_{aa} = \mathrm{id}_{T_a}.$$

Problems:

- How can we obtain a canonical enumeration of the basic open sets?
- How can we define a strong inclusion relation $\prec_\infty$ in terms of $\prec_a$?

Using only 1-1 continuous maps as embeddings gives not enough information!

Consider the case of algebraic domain again:

$$D \underset{p}{\overset{e}{\rightleftarrows}} D',$$

where

$$p \circ e = \mathrm{id}_D, \qquad e \circ p \sqsubseteq \mathrm{id}_{D'} .$$

It follows that

$$z \in D \text{ is finite } \implies e(z) \in D' \text{ is finite,}$$

i.e.,

$$\{\, y \in D \mid z \sqsubseteq y \,\} \rightsquigarrow \{\, u \in D' \mid e(z) \sqsubseteq' u \,\}.$$

Note that these sets are basic open.

Definition

Let $\mathcal{T}_1, \mathcal{T}_2$ be effectively given T_0 spaces. $(e, \hat{e})\colon \mathcal{T}_1 \to \mathcal{T}_2$ is an *embedding* if

- $e\colon T_1 \to T_2$ effectively continuous, 1-1,
- $\hat{e}\colon \mathcal{B}_1 \to \mathcal{B}_2$ effective, i.e.
  there is some computable function $\overline{e}$ with $\hat{e}(B^1_n) = B^2_{\overline{e}(n)}$,
- $e^{-1}[\hat{e}(B^1_n)] = B^1_n$,
- $(\forall O \in \tau_2) e[B^1_n] \subseteq O \Longrightarrow \hat{e}(B^1_n) \subseteq O$,
- $m \prec_1 m \Longleftrightarrow \overline{e}(m) \prec_2 \overline{e}(n)$,
- $B^1_m \cap B^1_n = \emptyset \Longrightarrow \hat{e}(B^1_m) \cap \hat{e}(B^1_n) = \emptyset$.

Definition

An open set $O \in \tau$ is *Lacombe open* if there is some semi-decidable set $A \subseteq \omega$ so that

$$O = \bigcup \{ B_n \mid n \in A \}.$$

Any r.e. index of A is a *Lacombe index* of O.

This gives rise to an indexing L^τ of the Lacombe open sets.

Definition

A map $f \colon T \to T'$ is *effectively continuous* if there is a computable function $\overline{f}$ so that

$$f^{-1}(B'_n) = L^\tau_{\overline{f}(n)}.$$

It follows that the notion of embedding is actually a point-free notion.

Note.

- $e[B^1_n] \subseteq \hat{e}(B^1_n)$.
- e is monotone with respect to the specialization orders $\leq_{\tau_1}$ and $\leq_{\tau_2}$.

Let $\mathbf{TOP}^{\mathbf{e}}_{\mathbf{eff}}$ with

- objects: effectively given T_0 spaces
- morphisms: embeddings.

Consider

$$\mathcal{T}^0 \underset{(e_0,\hat{e}_0)}{\longrightarrow} \mathcal{T}^1 \underset{(e_1,\hat{e}_1)}{\longrightarrow} \mathcal{T}^2 \longrightarrow \cdots$$

and take again

$$T^\infty = \bigoplus_{a\in\omega} T^a / \sim$$

with the canonical topology.

Lemma

For all $a, n \geq 0$,

$$\bigcup_{c<a} \{c\} \times c_{ca}^{-1}[B_n^a] \cup \{a\} \times B_n^a \cup \bigcup_{a<c} \{c\} \times \hat{e}_{ac}(B_n^a)$$

is open in $\bigoplus_{a\in\omega} T^a$ and closed with respect to $\sim$.

Let $\pi_\sim$ be the canonical projection associated with $\sim$ and set

$$B^\infty_{\langle a,n\rangle} = \pi_\sim\Big(\bigcup_{c<a}\{c\}\times c_{ca}^{-1}[B^a_n]\cup\{a\}\times B^a_n\cup\bigcup_{a<c}\{c\}\times \hat{e}_{ac}(B^a_n)\Big).$$

Lemma

$$B^\infty_{\langle a,n\rangle}\subseteq B^\infty_{\langle c,m\rangle}\iff [a\leq c\wedge \hat{e}_{ac}(B^a_n)\subseteq B^c_m]\vee[c<a\wedge B^a_n\subseteq \hat{e}_{ca}(B^c_m)].$$

Definition

$$\langle a,n\rangle \prec_\infty \langle c,m\rangle \iff [a\leq c\wedge \overline{e}_{ac}(n)\prec_c m]\vee[c<a\wedge n\prec_a \overline{e}_{ca}(m)].$$

Call the chain $\big(\mathcal{T}^a,(e_a,\hat{e}_a)\big)_{a\in\omega}$ *effective* if the effectivity requirements for $\mathcal{T}^a$ and $(e_a,\hat{e}_a)$ hold uniformly in a.

Theorem

Let $\left(\mathcal{T}^a, (e_a, \hat{e}_a)\right)_{a\in\omega}$ *be an effective chain in* $\mathbf{TOP}^{\mathbf{e}}_{\mathbf{eff}}$. *Then*

- $\mathcal{T}^\infty = (T^\infty, \tau^\infty, \mathcal{B}^\infty, \prec_\infty)$ *is effectively given.*
- $\mathcal{T}^\infty$ *is a colimit of* $\left(\mathcal{T}^a, (e_a, \hat{e}_a)\right)_{a\in\omega}$.

Initial Object in $\mathbf{TOP}^{\mathbf{e}}_{\mathbf{eff}}$:

Without restriction assume for $\mathcal{T} \in \mathbf{TOP}^{\mathbf{e}}_{\mathbf{eff}}$ that

$$\emptyset \in \mathcal{B}, \qquad B_0 = \emptyset, \qquad 0 \prec_B 0.$$

Set

$$B^{\emptyset}_n = \emptyset \ (n \in \omega) \quad \text{and} \quad \prec_{\emptyset} = \omega \times \omega.$$

Then

$$(\emptyset, \{\emptyset\}, \{\emptyset\}, \prec_{\emptyset}) \in \mathbf{TOP}^{\mathbf{e}}_{\mathbf{eff}}.$$

Let $(T, \tau, \mathcal{B}, \prec_B) \in \mathbf{TOP}^{\mathbf{e}}_{\mathbf{eff}}$. Then

$$\emptyset \underset{(\emptyset, (\emptyset \mapsto \emptyset))}{\longrightarrow} T,$$

where $(\emptyset, (\emptyset \mapsto \emptyset))$ is an embedding.

It follows that $(\emptyset, \{\emptyset\}, \{\emptyset\}, \prec_{\emptyset})$ is initial in $\mathbf{TOP}^{\mathbf{e}}_{\mathbf{eff}}$.

Call a functor $F\colon \mathbf{TOP}^{\mathbf{e}}_{\mathbf{eff}} \to \mathbf{TOP}^{\mathbf{e}}_{\mathbf{eff}}$ *effective*, if there are computable functions $\mathrm{ob}, \mathrm{mor}_1, \mathrm{mor}_2$ such that:

- If $i \in \omega$ is an r.e. index of $\prec_{B^T}$, then $\mathrm{ob}(i)$ is an r.e. index of $\prec_{B^{F(T)}}$.
- If i is a Gödel number of the function witnessing that e is effectively continuous, then $\mathrm{mor}_1(i)$ is a Gödel number of the function witnessing that $F(e)$ is effectively continuous.
- If i is a Gödel number of the function witnessing that $\hat{e}$ is effective, then $\mathrm{mor}_2(i)$ is a Gödel number of the function witnessing that $F(\hat{e})$ is effective.

Let $F\colon \mathbf{TOP}^{\mathbf{e}}_{\mathbf{eff}} \to \mathbf{TOP}^{\mathbf{e}}_{\mathbf{eff}}$ be effective. Then the chain

$$\emptyset \underset{(\emptyset,(\emptyset \mapsto \emptyset))}{\longrightarrow} F(\emptyset) \underset{F(\emptyset,(\emptyset \mapsto \emptyset))}{\longrightarrow} F^2(\emptyset) \longrightarrow \cdots$$

is effective. Hence it has a colimit $\mathcal{T}^F \in \mathbf{TOP}^{\mathbf{e}}_{\mathbf{eff}}$. With the result of Smyth and Plotkin we therefore have:

Theorem

Let $F\colon \mathbf{TOP}^{\mathbf{e}}_{\mathbf{eff}} \to \mathbf{TOP}^{\mathbf{e}}_{\mathbf{eff}}$ be effective and ω-continuous. Then

$$\mathcal{T}^F = F(\mathcal{T}^F).$$

Application:

Let (T, τ) be a finite T_0 space, fix an enumeration B of a topological basis $\mathcal{B}$, set

$$m \prec_B n \iff B_m \subseteq B_n.$$

Then

- $(T, \tau, \mathcal{B}, \prec_B)$ is effectively given.
- (Uniform) chains of finite T_0 spaces have a colimit in $\mathbf{TOP^e_{eff}}$.