Every Δ^0_2-Set is Natural, up to Turing Equivalence

Dieter Spreen

Lab. for Theoretical Computer Science, Dept. of Mathematics
University of Siegen, 57068 Siegen, Germany
spreen@math.uni-siegen.de

Abstract. The Turing degrees of Δ^0_2 sets as well as the degrees of sets in certain upper segments of the Ershov hierarchy are characterized. In particular, it is shown that, up to Turing equivalence and admissible indexing, the noncomputable Δ^0_2 sets are exactly the decision problems of countable partial orders of tree shape with branches of positive finite length only.

1 Introduction

Call a set of non-negative integers *natural* if it encodes a decision problem of some mathematical structure. In this note we characterize the Turing degrees of Δ^0_2 sets as well as the degrees of sets in certain upper segments of the Ershov hierarchy. Among others, we show that, up to Turing equivalence and admissible indexing, the noncomputable Δ^0_2 sets are exactly the decision problems of countable partial orders of tree shape with branches of positive finite length only.

Partially ordered sets of this kind are special Scott-Ershov domains [9]. These structures have been used with great success in logic and theoretical computer science, e.g. in higher-type computability and computing with exact real numbers [3, 6]. Each such domain contains a set of base elements from which all its members can be obtained by taking directed suprema. In applications the base elements can easily be encoded in such a way that the underlying order is semi-decidable. This allows to construct an *admissible* numbering of the constructive domain elements. From each admissible index of a constructive member one can uniformly compute an effective listing of the directed set of base elements lower or equal to the given member. Conversely, there is a procedure which given an effective enumeration of a directed set of base elements computes an admissible index of its least upper bound. Admissible numberings generalize the well-known Gödel numberings for the computable functions. So, they are highly not one-to-one.

In the last years there has been large interest in studying computable structures that have a copy in every degree of a given collection of Turing degrees [2, 7, 10]. In these cases the structures are supposed to be coded in an one-to-one way so that, up to isomorphism, their universe can be taken as a subset of ω.

The note is organized as follows: Sect. 2 contains basic definitions from computability and numbering theory. Constructive domains are introduced in Sect. 3, where also the above-mentioned results are derived.

2 Basic Definitions

In what follows, let $\langle\ ,\ \rangle\colon \omega^2 \to \omega$ be a computable pairing function. We extend the pairing function in the usual way to an n-tuple encoding. Let $R^{(n)}$ denote the set of all n-ary total computable functions and W_i be the domain of the ith partial computable function φ_i with respect to some Gödel numbering φ. We let $\varphi_i(a){\downarrow}_n$ mean that the computation of $\varphi_i(a)$ stops within n steps. In the opposite case we write $\varphi_i(a){\uparrow}_n$.

For sets $A, B \subseteq \omega$, we write $A \leq_T B$ if A is Turing reducible to B, and $A \equiv_T B$ in case A is Turing equivalent to B. The cardinality of A is denoted by $|A|$ and c_A is its characteristic function.

Let S be a nonempty set. A *numbering* ν of S is a map $\nu\colon \omega \to T$(onto). The value of ν at $n \in \omega$ is denoted, interchangeably, by ν_n and $\nu(n)$.

Definition 1. *Let ν and κ be numberings of sets S and S', respectively.*

1. *$\nu \leq \kappa$, read ν is* reducible *to κ, if there is a function $g \in R^{(1)}$ such that $\nu_m = \kappa_{g(m)}$, for all $m \in \omega$.*
2. *$\nu \leq_1 \kappa$, read ν is* one-one reducible *to κ, if $\nu \leq \kappa$ and the function g is one-to-one.*
3. *$\nu \equiv_1 \kappa$, read ν is* one-one equivalent *to κ, if $\nu \leq_1 \kappa$ and $\kappa \leq_1 \nu$.*

Obviously, if $\nu \leq \kappa$ then $S \subseteq S'$.

Definition 2. *A numbering ν of S is said to be*

1. decidable *if $\{ \langle n, m\rangle \mid n, m \in \omega \wedge \nu_n = \nu_m \}$ is recursive*
2. *a* cylinder *if $\nu \equiv_1 c(\nu)$, where $c(\nu)_{\langle n,m\rangle} = \nu_n$, for $n, m \in \omega$, is the* cylindrification *of ν.*

For a subset X of S^n, let $\Omega_\nu(X) = \{ \langle i_1, \ldots, i_n\rangle \mid (\nu_{i_1}, \ldots, \nu_{i_n}) \in X \}$. Then X is *completely enumerable*, if $\Omega_\nu(X)$ is computably enumerable. Similarly, X is *completely recursive* if $\Omega_\nu(X)$ is computable.

3 Constructive domains

Let $\mathcal{Q} = (Q, \sqsubseteq)$ be a partial order with smallest element $\bot$. A nonempty subset S of Q is *directed* if for all y_1, $y_2 \in S$ there is some $u \in S$ with y_1, $y_2 \sqsubseteq u$. $\mathcal{Q}$ is *directed-complete* if every directed subset S of Q has a least upper bound $\bigsqcup S$ in Q.

An element z of a directed-complete partial order $\mathcal{Q}$ is *finite* (or *compact*) if for any directed subset S of Q the relation $z \sqsubseteq \bigsqcup S$ always implies the existence of a $u \in S$ with $z \sqsubseteq u$. Denote the set of all finite members of Q by $\mathcal{K}$. If for any $y \in Q$ the set $\mathcal{K}_y = \{ z \in \mathcal{K} \mid z \sqsubseteq y \}$ is directed and $y = \bigsqcup \mathcal{K}_y$, $\mathcal{Q}$ is said to be *algebraic* or a *domain*. Moreover in this case, we call $\mathcal{Q}$ a *tree domain*, or simply a *tree*, if the diagram of the partial order is a tree.

In this note we will particularly be interested in domains consisting of finite elements only. We refer to such domains as *finitary*. Note that in this case each ascending chain is finite.

A domain $\mathcal{Q}$ is said to be *effectively given*, if there is an indexing β of $\mathcal{K}$, called the *canonical indexing*, such that the restriction of the domain order $\sqsubseteq$ to $\mathcal{K}$ is completely enumerable. In this case a member $y \in Q$ is called *computable* if $\mathcal{K}_y$ is completely enumerable. Let Q_c denote the set of all computable elements of Q, then $\mathcal{Q}_c = (Q_c, \sqsubseteq, \beta)$ is called *constructive domain*. We say that $\mathcal{Q}_c$ is a *recursive domain* if $\{ \langle i, j \rangle \mid \beta_i \sqsubseteq \beta_j \}$ is even computable. Obviously, β is decidable in this case.

A numbering η of the computable domain elements is *admissible* if $\{ \langle i, j \rangle \mid \beta_i \sqsubseteq \eta_j \}$ is computably enumerable and there is a function $d \in R^{(1)}$ with $\eta_{d(i)} = \bigsqcup \beta(W_i)$, for all indices $i \in \omega$ such that $\beta(W_i)$ is directed. As is shown in [11] such numberings exist. Moreover, $\beta \leq \eta$. For what follows we always assume that $\mathcal{Q}_c$ is a constructive domain indexed by an admissible numbering η.We will fix a uniform way of approximating the computable domain elements.

Lemma 1 ([8]). *Let β be decidable. Then there is a function* $\mathrm{en} \in R^{(1)}$ *such that for all $i \in \omega$,* $\mathrm{dom}(\varphi_{\mathrm{en}(i)})$ *is an initial segment of ω and the sequence of all $\beta(\varphi_{\mathrm{en}(i)}(c))$ with $c \in \mathrm{dom}(\varphi_{\mathrm{en}(i)})$ is strictly increasing with least upper bound η_i.*

In what follows let $\mathrm{ld}(i)$ be the cardinality of $\mathrm{dom}(\varphi_{\mathrm{en}(i)})$.

In [8] the difficulty of the problem to decide for two computable domain elements x, y whether $x \sqsubseteq y$ was studied in a general topological setting. As was shown, $\Omega_\eta(\sqsubseteq) \in \Pi^0_2$. The more exact localization depends on whether the constructive domain contains nonfinite elements or not.

Proposition 1 ([8]). *If Q_c contains a nonfinite element then $\Omega_\eta(\sqsubseteq)$ is Π^0_2-complete.*

Proposition 2 ([8]). *If Q_c contains only finite elements and the canonical indexing is decidable, then $\Omega_\eta(\sqsubseteq) \in \Delta^0_2$.*

Proof. We have to show that $\Omega_\eta(\sqsubseteq)$ is identified in the limit by some function $g \in R^{(2)}$. In what follows we construct a function g which is different from the one presented in [8] and which will be used again later. Let the function en be as in Lemma 1. Since Q_c is finitary, $\varphi_{\mathrm{en}(i)}$ is a finite function, for all $i \in \omega$. By admissibility of η there is a function $v \in R^{(1)}$ with $W_{v(i)} = \{\, a \mid \beta_a \sqsubseteq \eta_i \,\}$. Define $g, k \in R^{(2)}$ by the following simultaneous recursion:

$$k(\langle i,j\rangle, 0) = 0, \qquad g(\langle i,j\rangle, 0) = 1,$$

$$k(\langle i,j\rangle, s+1) = \begin{cases} k(\langle i,j\rangle, s) + 1 & \text{if } \varphi_{\mathrm{en}(i)}(k(\langle i,j\rangle, s))\downarrow_{s+1} \\ & \text{and } g(\langle i,j\rangle, s) = 1, \\ k(\langle i,j\rangle, s) & \text{otherwise,} \end{cases}$$

$$g(\langle i,j\rangle, s+1) = \begin{cases} 0 & \text{if } k(\langle i,j\rangle, s+1) \neq k(\langle i,j\rangle, s), \\ 1 & \text{if } \varphi_{v(j)}(\varphi_{\mathrm{en}(i)}(c))\downarrow_{s+1}, \text{ for all} \\ & c < k(\langle i,j\rangle, s), \text{ and } g(\langle i,j\rangle, s) = 0, \\ g(\langle i,j\rangle, s) & \text{otherwise.} \end{cases}$$

Then it follows for $i, j \in \omega$ that

$$\begin{aligned} \eta_i \sqsubseteq \eta_j &\Rightarrow (\forall a \in \mathrm{dom}(\varphi_{\mathrm{en}(i)}))\beta(\varphi_{\mathrm{en}(i)}(a)) \sqsubseteq \eta_j \\ &\Rightarrow (\exists s)(\forall a < \mathrm{ld}(i))\varphi_{\mathrm{en}(i)}(a)\downarrow_s \wedge \varphi_{v(j)}(\varphi_{\mathrm{en}(i)}(a))\downarrow_s \\ &\Rightarrow (\exists s)k(\langle i,j\rangle, s) = \mathrm{ld}(i) \wedge g(\langle i,j\rangle, s) = 1 \\ &\Rightarrow (\exists s)(\forall s' \geq s)g(\langle i,j\rangle, s') = 1 \\ &\Rightarrow \lim_s g(\langle i,j\rangle, s) = 1 \end{aligned}$$

and

$$\begin{aligned} \eta_i \not\sqsubseteq \eta_j &\Rightarrow (\exists a \in \mathrm{dom}(\varphi_{\mathrm{en}(i)}))\beta(\varphi_{\mathrm{en}(i)}(a)) \not\sqsubseteq \eta_j \\ &\Rightarrow (\exists a < \mathrm{ld}(i))[(\exists s)\varphi_{\mathrm{en}(i)}(a)\downarrow_s \wedge (\forall s')\varphi_{v(j)}(\varphi_{\mathrm{en}(i)}(a))\uparrow_{s'}] \\ &\Rightarrow (\exists a < \mathrm{ld}(i))[(\exists s)k(\langle i,j\rangle, s) = a + 1 \wedge g(\langle i,j\rangle, s) = 0 \\ &\qquad \wedge (\forall s')\varphi_{v(j)}(\varphi_{\mathrm{en}(i)}(a))\uparrow_{s'}] \\ &\Rightarrow (\exists s)(\forall s' \geq s)g(\langle i,j\rangle, s') = 0 \\ &\Rightarrow \lim_s g(\langle i,j\rangle, s) = 0. \qquad \square \end{aligned}$$

As a consequence of the above construction we have for all $i \in \omega$ that

$$\begin{aligned} \mathrm{ld}(i) &\leq |\{\, s \mid g(\langle i,i\rangle, s) \neq g(\langle i,i\rangle, s+1) \,\}|, \\ |\{\, s \mid g(\langle i,j\rangle, s) \neq g(\langle i,j\rangle, s+1) \,\}| &\leq 2\,\mathrm{ld}(i). \end{aligned} \tag{1}$$

The next result shows that a better localization than in the preceding proposition cannot be obtained without further restrictions. If $\mathcal{Q}_c$ is finitary the order decision problem can be at least as difficult as any Δ^0_2 problem.

Proposition 3 ([8]). *For any $A \in \Delta^0_2$ a finitary recursive tree domain can be be designed such that $A \leq_1 \Omega_\eta(\sqsubseteq)$.*

Note that a finitary tree is exactly one with branches of finite length only. We delineate the construction again as it will be used in the next steps.

Proof. Let $A \in \Delta^0_2$. Then there is a 0-1-valued function $g \in R^{(2)}$ such that $c_A(a) = \lim_s g(a,s)$. The idea is to construct a domain consisting of infinitely many chains which are glued together at their first element such that the length of the ath chain is the number of how often $\lambda s.g(a,s)$ changes its hypothesis enlarged by one, i.e.,

$$1 + |\{\, s \mid g(a,s) \neq g(a,s+1) \,\}|.$$

Let to this end $\bot \notin \omega$ and define

$$\begin{aligned} &Q_A \\ &= \{\bot\} \cup \{\, \langle a,s\rangle \mid [s=0 \wedge g(a,s)=1] \vee [s \neq 0 \wedge g(a,s-1) \neq g(a,s)] \,\}. \end{aligned}$$

Moreover, for $y, z \in Q^A$ let

$$y \sqsubseteq_A z \Leftrightarrow y = \bot \vee (\exists a, s, s' \in \omega) y = \langle a,s\rangle \wedge z = \langle a,s'\rangle \wedge s \leq s'.$$

Then $(Q_A, \sqsubseteq_A)$ is a tree domain. Set

$$\beta^A_{\langle a,s\rangle} = \begin{cases} \bot & \text{if } (\forall s' \leq s) g(a,s') = 0, \\ \langle a, \mu s' \leq s : (\forall s' \leq s'' \leq s) g(a,s'') = g(a,s)\rangle & \text{otherwise.} \end{cases}$$

Obviously, β^A is an indexing of Q_A with respect to which the partial order $\sqsubseteq_A$ is completely recursive. In this partial order every element has only finitely many predecessors and is thus finite as well as computable. It follows that $(Q_A, \sqsubseteq_A, \beta^A)$ is a finitary recursive domain.

Now, let η be an admissible numbering of Q^A, and let $h, k \in R^{(1)}$ with

$$W_{h(a)} = \{ \langle a, s \rangle \mid s \in \omega \} \text{ and } W_{k(a)} = \{ \langle a, s \rangle \mid g(a, s) = 1 \}.$$

Furthermore, let the function $d \in R^{(1)}$ be as in the definition of admissibility. Since $W_{k(a)} \subseteq W_{h(a)}$, we always have that $x_{d(k(a))} \sqsubseteq x_{d(h(a))}$. Let t_a be the smallest step s such that $g(a, s') = g(a, s)$, for all $s' \geq s$. Then it holds that $x_{d(k(a))} = x_{d(h(a))}$ if and only if $g(a, t_a) = 1$. With this we obtain

$$a \in A \Leftrightarrow g(a, t_a) = 1 \Leftrightarrow x_{d(h(a))} = x_{d(k(a))} \Leftrightarrow x_{d(h(a))} \sqsubseteq_A x_{d(k(a))}.$$

As shown in [11], admissible indexings are cylinders. The same is true for the numbering W. Thus, the functions d, h and k can be chosen as one-to-one. It follows that $A \leq_1 \Omega_\eta(\sqsubseteq_A)$. □

With respect to the weaker Turing equivalence, $\Omega_\eta(\sqsubseteq_A)$ turns out to be even as difficult as A.

Proposition 4. *For $A \in \Delta^0_2$, $\Omega_\eta(\sqsubseteq_A) \leq_T A$.*

Proof. Let $A \in \Delta^0_2$ and $g \in R^{(2)}$ such that $c_A(a) = \lim_s g(a, s)$. Moreover, be $r \in R^{(1)}$ with $W_{r(i)} = \text{range}(\varphi_i)$, $d \in R^{(1)}$ as in the definition of admissibility, and witness $k \in R^{(1)}$ that $\beta^A \leq \eta$.

For any $a \in \omega$, $\bigsqcup_s \beta^A_{\langle a,s \rangle}$ is the maximal element of the branch given by

$$\{\bot\} \cup \{ \langle a, s \rangle \mid [s = 0 \wedge g(a, s) = 1] \vee [s \neq 0 \wedge g(a, s-1) \neq g(a, s)] \}.$$

Let $h \in R^{(1)}$ with $\varphi_{h(a)}(s) = \langle a, s \rangle$ and $\bar{h} = r \circ h$. Then $W_{\bar{h}(a)} = \{ \langle a, s \rangle \mid s \in \omega \}$ and hence

$$\bigsqcup\nolimits_s \beta^A_{\langle a,s \rangle} = \bigsqcup \beta^A(W_{\bar{h}(a)}) = \eta_{d(\bar{h}(a))}.$$

As follows from the construction of $\mathcal{Q}_A$, $c_A(a) = g(a, \mu s : \beta^A_{\langle a,s \rangle} = \eta_{d(\bar{h}(a))})$, i.e.,

$$c_A(a) = g(a, \mu s : \langle k(\langle a, s \rangle), d(\bar{h}(a)) \rangle \in \Omega_\eta(=)).$$

Note that $\Omega_\eta(=) = \{ \langle i, j \rangle \mid \langle i, j \rangle, \langle j, i \rangle \in \Omega_\eta(\sqsubseteq_A) \}$. Thus we have that $A \leq_T \Omega_\eta(\sqsubseteq_A)$. □

Summarizing what we have seen so far, we obtain the first of our main results.

Theorem 1. *For Turing degrees* **a** *the following three statements are equivalent:*

1. $\mathbf{a} = \deg_T(A)$, *for some set* $A \in \Delta_2^0$.
2. $\mathbf{a} = \deg_T(\Omega_\eta(\sqsubseteq))$, *for some finitary recursive tree.*
3. $\mathbf{a} = \deg_T(\Omega_\eta(\sqsubseteq))$, *for some finitary constructive domain with decidable canonical indexing.*

As is well-known, the class of Δ_2^0 sets is exhausted by the Ershov hierarchy $\{\, \Sigma_\alpha^{-1}, \Pi_\alpha^{-1} \mid \alpha \text{ an ordinal} \,\}$ (also known as the Boolean hierarchy) [4, 5, 1]. Let K denote the halting problem and $\overline{K}$ its complement. For $m \geq 1$ set

$$Z_m = \bigcup_{i=0}^{m} (K^{2i} \times \overline{K}^{2(m-i)})$$

and let

$$Z_\omega = \{\, \langle n, i \rangle \mid i \in Z_n \wedge n \geq 1 \,\}.$$

Then Z_m and Z_ω, respectively, are m-complete for Π_{2m}^{-1} and Π_ω^{-1}. In certain cases also the order decision problem is m-complete for the corresponding levels of the hierarchy.

Let $\mathcal{Q}$ be finitary and CH be the set of all lengths of increasing chains in Q. In general, this collection will be unbounded. Let us first consider the bounded case. Set $\mathrm{lc} = \max \mathrm{CH}$.

Proposition 5 ([8]). *Let* $\mathcal{Q}_c$ *be finitary so that* CH *is bounded and let the canonical indexing be decidable. If* $\mathrm{lc} \geq 1$ *then* $\Omega_\eta(\sqsubseteq)$ *is* m*-complete for* $\Pi_{2\,\mathrm{lc}}^{-1}$. *Otherwise,* $\Omega_\eta(\sqsubseteq)$ *is computable.*

Let us now turn to the unbounded case.

Proposition 6 ([8]). *Let* $\mathcal{Q}_c$ *be finitary so that* CH *is unbounded and let the canonical indexing be decidable. Then* $Z_\omega \leq_m \Omega_\eta(\sqsubseteq)$.

In the above theorem we have seen that the Turing degrees of the order decision problem of finitary constructive domains that come with a decidable canonical indexing, as well as the Turing degrees of the order decision problem of finitary recursive trees, capture exactly the Δ_2^0 degrees. With the preceding result we are now able to capture exactly the Turing degrees of sets in certain upper segments of Δ_2^0.

Theorem 2. *For* $m \in \omega$ *and Turing degrees* **a** *the following three statements are equivalent:*

1. $\mathbf{a} = \deg_T(A)$, *for some* $A \in \Delta_2^0 \setminus \Pi_m^{-1}$.
2. $\mathbf{a} = \deg_T(\Omega_\eta(\sqsubseteq))$, *for some finitary recursive tree such that either* CH *is unbounded or* CH *is bounded with* $2\,\mathrm{lc} > m$.
3. $\mathbf{a} = \deg_T(\Omega_\eta(\sqsubseteq))$, *for some finitary constructive domains with decidable canonical indexing such that either* CH *is unbounded or* CH *is bounded with* $2\,\mathrm{lc} > m$.

Proof. ($1 \Rightarrow 2$) Let $A \in \Delta_2^0 \setminus \Pi_m^{-1}$ and $\mathcal{Q}_A$ be as in Proposition 3. Then $A \leq_1 \Omega_\eta(\sqsubseteq_A)$. Assume that CH is bounded with $2\,\mathrm{lc} \leq m$. By Proposition 5 it follows that $\Omega_\eta(\sqsubseteq_A) \in \Pi_{2\,\mathrm{lc}}^{-1} \subseteq \Pi_m^{-1}$, which implies that also $A \in \Pi_m^{-1}$, a contradiction.

The implication ($2 \Rightarrow 3$) is obvious. For ($3 \Rightarrow 1$) let $\mathcal{Q}_c$ be a finitary constructive domain with decidable canonical indexing. If CH is unbounded, we have that $Z_\omega \leq_m \Omega_\eta(\sqsubseteq)$. Hence $\Omega_\eta(\sqsubseteq) \notin \Pi_m^{-1}$, as otherwise we would have that $Z_\omega \in \Pi_m^{-1}$ and hence that $\Pi_\omega^{-1} = \Pi_m^{-1}$.

If CH is bounded with $2\,\mathrm{lc} > m$, $\Omega_\eta(\sqsubseteq)$ is m-complete for $\Pi_{2\,\mathrm{lc}}^{-1}$, from which we obtain that $\Omega_\eta(\sqsubseteq) \notin \Pi_m^{-1}$. □

Note that for a finitary domain $\mathcal{Q}$, CH is unbounded or bounded with $\mathrm{lc} > 0$, exactly if Q has at least two elements. This leads to the following consequence.

Corollary 1. *For Turing degrees* $\mathbf{a}$ *the following three statements are equivalent:*

1. $\mathbf{a}$ *is the Turing degree of a noncomputable set in* Δ_2^0.
2. $\mathbf{a}$ *is the degree* $\deg_T(\Omega_\eta(\sqsubseteq))$ *of the order decision problem for some finitary recursive tree with branches of positive finite length only.*
3. $\mathbf{a}$ *is the degree* $\deg_T(\Omega_\eta(\sqsubseteq))$ *of the order decision problem for some finitary constructive domains of at least cardinality two with decidable canonical indexing*

As we have seen in Proposition 6, if Q_c is finitary and CH unbounded then $Z_\omega \leq_m \Omega_\eta(\sqsubseteq)$. In the tree domain $\mathcal{Q}_A$ constructed in the proof of Proposition 3 the length of the branches is determined by the number of mind changes of the function identifying the given problem A in the limit. Let us now consider the case that this function is recursively majorized.

Lemma 2. *Let* $C \in \Delta_2^0$ *be identified in the limit by the function* $g \in R^{(2)}$. *Then* $C \leq_m Z_\omega$, *precisely if* $\lambda a.|\{\, s \mid g(a,s) \neq g(a,s+1) \,\}|$ *is recursively majorized.*

Proof. The "if"-part is obvious and the "only if"-part shown in [8].

Now, let $\mathcal{Q}_c$ be a finitary constructive domain with decidable canonical indexing and $g \in R^{(2)}$ be the function constructed in the proof of Proposition 2 which identifies $\Omega_\eta(\sqsubseteq)$ in the limit. By the Inequalities (1) we have that $\lambda a.|\{\, s \mid g(a, s) \neq g(a, s + 1)\,\}|$ is recursively majorized, exactly if ld is.

This gives us the following characterization of the Turing degrees of sets in $\Delta^0_2 \setminus \Pi^{-1}_\omega$.

Theorem 3. *For Turing degrees* **a** *the following three statements are equivalent:*

1. $\mathbf{a} = \deg_T(A)$*, for some set* $A \in \Delta^0_2 \setminus \Pi^{-1}_\omega$.
2. $\mathbf{a} = \deg_T(\Omega_\eta(\sqsubseteq))$*, for some finitary recursive tree such that* CH *is unbounded and* ld *not recursively majorized.*
3. $\mathbf{a} = \deg_T(\Omega_\eta(\sqsubseteq))$*, for some finitary constructive domain with decidable canonical indexing such that* CH *is unbounded and* ld *not recursively majorized.*

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