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A CONSTRUCTION METHOD FOR PARTIAL METRICS

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Abstract. We present a general construction that starts from a family of interior-preserving open coverings of a given subspace and results in a partial metric with respect to which all subspace elements have self-distance zero. A necessary and sufficient condition is derived for when this partial metric induces the given topology. The condition is particularly satisfied if the members of each covering are pairwise disjoint.

The method is based on Fletcher's universal construction for transitive quasi-uniformities. Important examples of partial metrics in the literature can be obtained in this way.

As a consequence of the construction, the set of all points with self-distance zero is a G_δ. Moreover, this subspace is zero-dimensional in its induced topology.

1. Introduction

Topological spaces that appear in the context of the semantics of computation usually satisfy only the weak T_0 separation axiom, and are thus equipped with a canonical partial order. In contrast to spaces which are mainly considered in mathematics and are at least Hausdorff, these spaces contain not only the *ideal* or *total* elements that are the result of a mathematical construction or a computation

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and that are the only points in commonly considered spaces, but also the objects which appear at the various computational stages. They are called *partial* elements and approximate the total ones.

Because of the just mentioned weak topological separation properties metrics are no longer appropriate when dealing with quantitative aspects of such approximations. One has to use quasi-metrics instead. They are such that the distance from point x to point y may be different to that from y to x. This seems unnatural and counter intuitive. As a remedy Matthews [15] introduced the notion of a partial metric. In this case the distance function is symmetric, but the self-distance of a point need not be zero. In applications one is mostly interested in partial metrics with respect to which the ideal elements have self-distance zero. The self-distance of the partial elements is interpreted as a measure of their partiality.

Each partial metric induces a quasi-metric in a natural way. In fact, partial metrics are equivalent to weighted quasi-metrics [15]. Their topology is the topology of the associated quasi-metric. As is well known, each second-countable T_0 space is quasi-metrizable. This does not hold for partial metrics. Künzi and Vajner [13] provide a subtle discussion of which spaces are partial metrizable.

Every quasi-metric generates a quasi-uniformity in the usual way. Conversely, every countably based quasi-uniformity with associated T_0 topology can be generated in such a way. It is not known whether this is also true for partial metrics.

In this note we present a general construction that starts from a family of interior-preserving open coverings of a given subspace and results in a partial metric with respect to which all subspace elements have self-distance zero. A necessary and sufficient condition is derived for when this partial metric induces the given topology. The condition is particularly satisfied if the members of each covering are pairwise disjoint.

The method is based on Fletcher's universal construction for transitive quasi-uniformities [6, 7]. Important examples of partial metrics in the literature can be obtained in this way.

As a consequence of the construction, the set of all points with self-distance zero is a G_δ. By definition, the partial metric is a metric on this subspace. Since the uniformity generated by it is transitive, the subspace is zero-dimensional in its induced topology.

The note is organized as follows: In Section 2 basic topological definitions and facts are recalled. Section 3 contains definitions and examples from domain theory. The construction is given in Section 4.

2. Weak Metrics

In this section we recall some topological notions and facts needed in what follows.

Let (X, ρ) be a topological space. If U is a subset of X then $\mathrm{ext}_\rho(U)$ denotes its exterior. For points $x, y \in X$ set $x \leq_\rho y$ if every open set in ρ containing x also contains y. Then $\leq_\rho$ is a preorder on X, called the *specialization order*. It is a partial order exactly if the space is T_0.

Definition 2.1. A *quasi-uniformity* on a set X is a filter $\mathcal{U}$ of binary relations on X such that

(1) each member of $\mathcal{U}$ contains the diagonal Δ_X of $X \times X$ and
(2) for any $U \in \mathcal{U}$ there is some $V \in \mathcal{U}$ with $V \circ V \subseteq U$.

The pair $(X,\mathcal{U})$ is said to be a *quasi-uniform space* and the elements of $\mathcal{U}$ are called *entourages*. A subfamily $\mathcal{B}$ of $\mathcal{U}$ is a *base* for $\mathcal{U}$ if each member of $\mathcal{U}$ contains an element of $\mathcal{B}$ and a subfamily $\mathcal{S}$ of $\mathcal{U}$ is a *subbase* for $\mathcal{U}$ if the family of finite intersections of elements of $\mathcal{S}$ is base for $\mathcal{U}$. A (sub)base $\mathcal{B}$ for a quasi-uniformity is *transitive* provided that each $B \in \mathcal{B}$ is a transitive relation. If a quasi-uniformity has a transitive (sub)base it is called *transitive* as well. For a detailed treatment of the theory of quasi-uniform spaces the reader is referred to [7].

If $\mathcal{U}$ is a quasi-uniformity on X, then so is its *conjugate* $\mathcal{U}^{-1} = \{ U^{-1} \mid U \in \mathcal{U} \}$, where $U^{-1} = \{ (x, y) \in X \times X \mid (y, x) \in U \}$. The uniformity $\mathcal{U}^*$ *generated* by $\mathcal{U}$ has a base given by the entourages $U^* = U \cap U^{-1}$. The topology $\tau(\mathcal{U})$ *induced* by $\mathcal{U}$ is that in which the sets $U(x) = \{ y \in X \mid (x, y) \in U \}$, with $U \in \mathcal{U}$, form a neighbourhood base for each $x \in X$. If ρ is a topology on X that coincides with $\tau(\mathcal{U})$, then $\mathcal{U}$ is said to be *compatible* with ρ.

As is well known, there is a close connection between uniformities and metrics on a set X. The same holds for quasi-uniformities and quasi-metrics.

Definition 2.2. A *quasi-metric* for a set X is a nonnegative real-valued function d on $X \times X$ such that the following two conditions hold for all $x, y, z \in X$:

(1) $x = y \Leftrightarrow d(x, y) = d(y, x) = 0$,
(2) $d(x, y) \le d(x, z) + d(z, y)$.

Each quasi-metric d on X *generates* a quasi-uniformity $\mathcal{U}_d$ on X which has as a base the family of sets of the form $\{\, (x, y) \in X \times X \mid d(x, y) < 2^{-n} \,\}$ with $n \in \mathbb{N}$.

For $x \in X$ and $n \in \mathbb{N}$, let $S_d(x, 2^{-n}) = \{\, y \in X \mid d(x, y) < 2^{-n} \,\}$ be the open sphere about x with radius 2^{-n}. The topology that has the family $\{\, S_d(x, 2^{-n}) \mid x \in X \wedge n \in \mathbb{N} \,\}$ as a base is called the topology *induced* by d and denoted by $\tau(d)$. It is identical with the topology $\tau(\mathcal{U}_d)$ induced by the quasi-uniformity generated by d.

As follows from the definition, a quasi-metric satisfies the usual conditions for a metric except the symmetry requirement. A distance notion closely related to quasi-metrics satisfying this condition is that of a partial metric.

Definition 2.3. A *partial metric* for a set X is a nonnegative real-valued function p on $X \times X$ such that the following four conditions are satisfied for any $x, y, z \in X$:

(1) $p(x, y) \ge p(x, x)$,
(2) $p(x, y) = p(y, x)$,
(3) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,
(4) $p(x, y) + p(z, z) \le p(x, z) + p(z, y)$.

The pair (X, p) is a *partial metric space*. The set $\ker(p) = \{\, x \in X \mid p(x, x) = 0 \,\}$ is called the *kernel* of p. For $x, y \in X$, let $x \le_p y$ if $p(x, y) = p(x, x)$. Then $\le_p$ is a partial order on X, said to be the partial order *associated* with p.

Proposition 2.4 ([15])**.** *Let (X, p) be a partial metric space. Then the kernel of p is upwards closed with respect to $\le_p$. All of its elements are even maximal with respect to this order.*

As has already been mentioned, spaces used in studies of the mathematical semantics of computation have only the weak T_0 separation property. In addition to the (totally defined) final results of a computation they also contain its intermediate values. The self-distance $p(x, x)$ is a feature used to describe the amount of information contained in x. The smaller $p(x, x)$ the more defined x is, x being totally defined if $p(x, x) = 0$.

Matthews [15] showed that there is a correspondence between partial metrics and weighted quasi-metrics, where a *weighted quasi-metric* on X is a pair of maps (d, w) consisting of a quasi-metric d on X and a nonnegative real-valued function w on X, called *weight function*, such that for all $x, y \in X$, $d(x, y) + w(x) = d(y, x) + w(y)$.

If p is a partial metric on X define the functions d_p and w_p, respectively, by

$$d_p(x, y) = p(x, y) - p(x, x) \text{ and}$$
$$w_p(x) = p(x, x).$$

Then (d_p, w_p) is a weighted quasi-metric on X. Conversely, if (d, w) is a weighted quasi-metric on X then the function $p_{(d,w)}$ defined by $p_{(d,w)}(x, y) = d(x, y) + w(x)$ is a partial metric on X.

We call d_p the quasi-metric *associated* with p and define the quasi-uniformity $\mathcal{U}_p$ *generated* and the topology $\tau(p)$ *induced* by the partial metric p, respectively to be the quasi-uniformity generated and the topology induced by the quasi-metric associated with p, i.e., $\mathcal{U}_p = \mathcal{U}_{d_p}$ and $\tau(p) = \tau(d_p)$. As is readily verified, the specialization order of $\tau(p)$ and the partial order associated with p agree. If ρ is a topology on X which coincides with the topology $\tau(p)$ induced by the partial metric p, then we also say that p is *compatible* with ρ.

3. Examples from Domain Theory

Domains are the standard type of spaces used in computer science when dealing with the semantics of computations on non-discrete objects.

Let $(D, \sqsubseteq)$ be a partial order with least element $\perp$. For a subset S of D, $\uparrow S = \{ x \in D \mid (\exists y \in S) y \sqsubseteq x \}$ is the *upper* set generated by S. The subset S is *directed*, if it is nonempty and every pair of elements in S has an upper bound in S. D is a *directed-complete* partial order (dcpo) if every directed subset S of D has a least upper bound $\bigsqcup S$ in D. Standard references for the theory of this structure are [10, 9, 1, 19, 2, 8].

If $(D, \sqsubseteq)$ is a dcpo and $x, y \in D$ then one says that x *approximates* y, and writes $x \ll y$ if for every directed subset S of D

with $y \sqsubseteq \bigsqcup S$ there is some $u \in S$ with $x \sqsubseteq u$. The relation $\ll$ is transitive. A member x of a dcpo D is *compact* if $x \ll x$. We denote the set of compact elements of D by K_D. Observe that for compact elements z and members x of D, $z \ll x$ exactly if $z \sqsubseteq x$.

Definition 3.1. A subset Z of a dcpo D is a *basis* for D, if for any $x \in D$ the set $Z_x = \{\, z \in Z \mid z \ll x \,\}$ is directed and $x = \bigsqcup Z_x$.

Note that K_D is included in any basis of D.

Definition 3.2. A dcpo D is called *continuous* or a *continuous domain* if it has a basis. It is called *algebraic* or *algebraic domain* if it has a basis of compact elements. We say that D is *ω-continuous* if there exists a countable basis and we call it *ω-algebraic* if K_D is a countable basis.

Several meaningful topologies have be defined on dcpo's. Mostly they are equipped with the Scott topology σ. It reflects important properties of computations. A subset X of D is *Scott open* if it is upwards closed with respect to the partial order and intersects each directed subset of which it contains the least upper bound. In case D is continuous with basis Z the family of sets $\twoheaduparrow\{z\} = \{\, x \in D \mid z \ll x \,\}$, with $z \in Z$, is a base for σ. Note that the Scott topology fulfills the T_0 but in general not the T_1 condition. Its specialization order coincides with the domain order.

Domain theory is a theory about the approximation by basis elements. Without any notion of distance only qualitative statements can be made. So, the question came up whether domains are partial metrizable [16, 11]. Positive answers were given in the ω-algebraic case by O'Neill [17], and in the more general ω-continuous case independently by Schellekens [18] and Waszkiewicz [21].

Let D be an ω-algebraic domain and $z_0, z_1, \dots$ an enumeration of its compact elements. For $x, y \in D$ set

$$p(x,y) = 1 - \sum \{\, 2^{-(n+1)} \mid n \geq 0 \wedge x, y \in \uparrow\{z_n\} \,\}. \tag{3.1}$$

Proposition 3.3 ([21]). *For any ω-algebraic domain D, p is a partial metric on D which is compatible with the Scott topology.*

Obviously, $p(x,x) = 0$ exactly if x dominates all elements of D. Since, in general, domains do not possess a greatest element, all elements have nonzero self-distance in this case.

Recent research in domain theory has mostly dealt with the question which spaces can be given a domain environment, i.e. a continuous domain which contains a subspace (of total elements) corresponding to the given space in a continuous way (cf. e.g. [3, 14, 5, 4, 12]). The ultimate goal of this research is the development of programming languages for real number computations that can be given a faithful domain semantics. In many cases the given space is required to be homeomorphic to the subspace of maximal domain elements. In this case one wants the partial metric to be such that its restriction to the maximal elements is a metric, or in other words, that exactly the maximal elements have self-distance zero. For spaces in which all maximal elements are also constructively maximal such partial metrics have been presented by Waszkiewicz [22] in the algebraic and Smyth [20] in the more general continuous case. For continuous domains in general the problem whether such partial metrics exist is still open.

An element x of a continuous domain D is *constructively maximal* if it satisfies all tests, where a *test* consists of a pair (u, v) of basis elements of D such that $u \ll v$, and x *satisfies* test (u, v) if either $u \ll x$ or x has a neighbourhood lying apart from $\Uparrow\{v\}$. In the algebraic case a test (u, v) can be simplified to the case $u = v$.

Let D be again an ω-algebraic domain and $z_0, z_1, \ldots$ an enumeration of its compact elements. For $x, y \in D$ set

$$\begin{aligned} q(x,y) = 1 - \sum \{\, 2^{-(n+1)} \mid n \geq 0 \wedge \\ [x, y \in \uparrow\{z_n\} \vee x, y \in \mathrm{ext}_\sigma(\uparrow\{z_n\})] \,\}. \end{aligned} \tag{3.2}$$

Proposition 3.4 ([20]). *For any ω-algebraic domain D, q is a partial metric on D which is compatible with the Scott topology. Moreover, $q(x, x) = 0$ exactly if x is constructively maximal.*

4. The Construction

Let (X, ρ) be a topological space and $\mathcal{C}$ a collection of open subsets of X. $\mathcal{C}$ is called *interior-preserving* if $\bigcap \{\, C \mid C \in \mathcal{C}' \,\}$ is open, for every $\mathcal{C}' \subseteq \mathcal{C}$. Set

$$U_\mathcal{C} = \{\, (x, y) \in X \times X \mid (\forall C \in \mathcal{C})[x \in C \Rightarrow y \in C] \,\}.$$

Then $U_\mathcal{C}$ is a reflexive and transitive relation.

The following result due to Fletcher [6, 7, Theorem 2.6] presents a universal construction method for transitive quasi-uniformities on X that are compatible with the given topology ρ.

Theorem 4.1. *Let (X, ρ) be a topological space. Then following two statements hold:*

(1) *Let $\mathfrak{A} = (\mathcal{C}^i)_{i\in I}$ be a family of interior-preserving collections of open subsets of X such that $\bigcup_{i\in I} \mathcal{C}^i$ is a subbase for ρ. Then $\{ U_{\mathcal{C}^i} \mid i \in I \}$ is a subbase for a transitive quasi-uniformity $\mathcal{U}_{\mathfrak{A}}$ on X which is compatible with ρ.*
(2) *Let $\mathcal{V}$ be a transitive quasi-uniformity on X compatible with ρ and let $\mathcal{B}$ be a transitive base for $\mathcal{V}$. For $V \in \mathcal{B}$ set $\mathcal{C}_V = \{ V(x) \mid x \in X \}$ and let $\mathfrak{B} = \{ \mathcal{C}_V \mid V \in \mathcal{B} \}$. Then $\mathfrak{B}$ is a family of interior-preserving collections of open covers of X such that $\bigcup \mathfrak{B}$ is a subbase for ρ and $\mathcal{V} = \mathcal{U}_{\mathfrak{B}}$.*

For the next steps the following two examples provide some inside.

Example 4.2. Let D be an algebraic domain. We will define two families of interior-preserving collections of open subsets of D. For $z \in K_D$ set

$$\mathcal{P}^z = \{\uparrow\{z\}\} \qquad \text{and} \qquad \mathcal{S}^z = \{\uparrow\{z\}, \mathrm{ext}_\sigma(\uparrow\{z\})\}.$$

Then $\bigcup \{ \mathcal{P}^z \mid z \in K_D \}$ and $\bigcup \{ \mathcal{S}^z \mid z \in K_D \}$ are subbases of σ. Moreover,

$$\begin{aligned} U_{\mathcal{P}^z} &= \uparrow\{z\} \times \uparrow\{z\} \cup \bigcup \{ \{x\} \times D \mid x \in D \backslash \uparrow\{z\} \} \qquad \text{and} \\ U_{\mathcal{S}^z} &= \uparrow\{z\} \times \uparrow\{z\} \cup \mathrm{ext}_\sigma(\uparrow\{z\}) \times \mathrm{ext}_\sigma(\uparrow\{z\}) \\ &\qquad \cup \bigcup \{ \{x\} \times D \mid x \in D \setminus (\uparrow\{z\} \cup \mathrm{ext}_\sigma(\uparrow\{z\})) \}. \end{aligned}$$

We see that both entourages have a significant symmetric subrelation, which is an equivalence relation on $\bigcup \mathcal{P}^z$ and/or $\bigcup \mathcal{S}^z$.

For an interior-preserving collection $\mathcal{C}$ of open subsets of X set

$$\mathrm{Sym}(U_{\mathcal{C}}) = U_{\mathcal{C}} \cap U_{\mathcal{C}}^{-1} \cap \bigcup \mathcal{C} \times \bigcup \mathcal{C}.$$

Then $\mathrm{Sym}(U_{\mathcal{C}})$ is an equivalence relation on $\bigcup \mathcal{C}$.

Assume now that (X, ρ) is a T_0 space and $\mathfrak{A} = (\mathcal{C}^i)_{i\in\mathbb{N}}$ is a family of interior-preserving collections of open subsets of X such that $\bigcup_{i\in\mathbb{N}} \mathcal{C}^i$ is a subbase for ρ. Let $x, y \in X$ and define

$$p_{\mathfrak{A}}(x,y) = 1 - \sum \{\, 2^{-(i+1)} \mid i \geq 0 \wedge (x,y) \in \mathrm{Sym}(U_{\mathcal{C}^i}) \,\}.$$

Proposition 4.3. *Let (X, ρ) be a T_0 space and $\mathfrak{A} = (\mathcal{C}^i)_{i\in\mathbb{N}}$ be a family of interior-preserving collections of open subsets of X such that $\bigcup_{i\in\mathbb{N}} \mathcal{C}^i$ is a subbase for ρ. Then the function $p_{\mathfrak{A}}$ is a partial metric on X.*

Proof. Condition 2.3(2) is obvious. For the verification of the remaining requirements let $x, y, z \in X$.

(1) If $(x,y) \in \mathrm{Sym}(U_{\mathcal{C}^i})$, then $x \in \bigcup \mathcal{C}^i$ and hence $(x,x) \in \mathrm{Sym}(U_{\mathcal{C}^i})$. Thus,

$$\begin{aligned} &\sum \{\, 2^{-(i+1)} \mid i \geq 0 \wedge (x,y) \in \mathrm{Sym}(U_{\mathcal{C}^i}) \,\} \\ &\qquad \leq \sum \{\, 2^{-(i+1)} \mid i \geq 0 \wedge (x,x) \in \mathrm{Sym}(U_{\mathcal{C}^i}) \,\}. \end{aligned}$$

It follows that $p_{\mathfrak{A}}(x,x) \leq p_{\mathfrak{A}}(x,y)$.

(3) If $p_{\mathfrak{A}}(x,y) = p_{\mathfrak{A}}(x,x)$, then

$$\begin{aligned} &\sum \{\, 2^{-(i+1)} \mid i \geq 0 \wedge (x,y) \in \mathrm{Sym}(U_{\mathcal{C}^i}) \,\} \\ &\qquad = \sum \{\, 2^{-(i+1)} \mid i \geq 0 \wedge (x,x) \in \mathrm{Sym}(U_{\mathcal{C}^i}) \,\} \\ &\qquad = \sum \{\, 2^{-(i+1)} \mid i \geq 0 \wedge x \in \textstyle\bigcup \mathcal{C}^i \,\}, \end{aligned}$$

which implies that for $i \geq 0$,

$$(x,y) \in \mathrm{Sym}(U_{\mathcal{C}^i}) \Leftrightarrow x \in \bigcup \mathcal{C}^i.$$

Now, let $C \in \bigcup_{i\in\mathbb{N}} \mathcal{C}^i$ with $x \in C$. Then there is some $n \geq 0$ so that $C \in \mathcal{C}^n$. By what we have just seen it follows that $(x,y) \in \mathrm{Sym}(U_{\mathcal{C}^n})$. Since $x \in C$ we have that $y \in C$ as well. Thus $x \leq_\rho y$, as $\bigcup_{i\in\mathbb{N}} \mathcal{C}^i$ is a subbase for ρ.

If also $p_{\mathfrak{A}}(x,y) = p_{\mathfrak{A}}(y,y)$, we obtain in the same way that $y \leq_\rho x$. It follows that $x = y$, since (X,ρ) is T_0.

(4) If $(x,z) \in \mathrm{Sym}(U_{\mathcal{C}^i})$ or $(z,y) \in \mathrm{Sym}(U_{\mathcal{C}^i})$, then we have in any case that $z \in \bigcup \mathcal{C}^i$ and $(z,z) \in \mathrm{Sym}(U_{\mathcal{C}^i})$. Moreover, if $(x,z), (z,y) \in \mathrm{Sym}(U_{\mathcal{C}^i})$, then also $(x,y) \in \mathrm{Sym}(U_{\mathcal{C}^i})$, as $\mathrm{Sym}(U_{\mathcal{C}^i})$

is transitive. Thus

$$\begin{aligned}
&\sum \{\, 2^{-(i+1)} \mid i \geq 0 \wedge (x,z) \in \mathrm{Sym}(U_{\mathcal{C}^i}) \,\} \\
&\qquad\qquad + \sum \{\, 2^{-(i+1)} \mid i \geq 0 \wedge (z,y) \in \mathrm{Sym}(U_{\mathcal{C}^i}) \,\} \\
&\leq \sum \{\, 2^{-(i+1)} \mid i \geq 0 \wedge (x,y) \in \mathrm{Sym}(U_{\mathcal{C}^i}) \,\} \\
&\qquad\qquad + \sum \{\, 2^{-(i+1)} \mid i \geq 0 \wedge (z,z) \in \mathrm{Sym}(U_{\mathcal{C}^i}) \,\}.
\end{aligned}$$

It follows that $p_{\mathfrak{A}}(x,y) + p_{\mathfrak{A}}(z,z) \leq p_{\mathfrak{A}}(x,z) + p_{\mathfrak{A}}(z,y)$.

$\square$

Moreover, we have for $x \in X$ that

$$\begin{aligned}
p_{\mathfrak{A}}(x,x) = 0 &\Leftrightarrow \sum \{\, 2^{-(i+1)} \mid i \geq 0 \wedge (x,x) \in \mathrm{Sym}(U_{\mathcal{C}^i}) \,\} = 1 \\
&\Leftrightarrow \sum \{\, 2^{-(i+1)} \mid i \geq 0 \wedge x \in \bigcup \mathcal{C}^i \,\} = 1 \\
&\Leftrightarrow (\forall i \in \mathbb{N}) x \in \bigcup \mathcal{C}^i \\
&\Leftrightarrow x \in \bigcap \{\, \bigcup \mathcal{C}^i \mid i \in \mathbb{N} \,\}.
\end{aligned}$$

Proposition 4.4. *Let (X,ρ) be a T_0 space and $\mathfrak{A} = (\mathcal{C}^i)_{i\in\mathbb{N}}$ be a family of interior-preserving collections of open subsets of X such that $\bigcup_{i\in I} \mathcal{C}^i$ is a subbase for ρ. Then the kernel of $p_{\mathfrak{A}}$ is a G_δ-set, and hence upwards closed with respect to the specialization order.*

As an immediate consequence we obtain that if S is a subspace of X and we choose the $\mathcal{C}^i$, for $i \in \mathbb{N}$, as interior-preserving open covers of S then S is included in the kernel of $p_{\mathfrak{A}}$.

Let $d_{\mathfrak{A}}$ be the quasi-metric associated with $p_{\mathfrak{A}}$. Then

$$\begin{aligned}
&d_{\mathfrak{A}}(x,y) \\
&\quad = p_{\mathfrak{A}}(x,y) - p_{\mathfrak{A}}(x,x) \\
&\quad = \sum \{\, 2^{-(i+1)} \mid i \geq 0 \wedge x \in \bigcup \mathcal{C}^i \wedge (x,y) \notin \mathrm{Sym}(U_{\mathcal{C}^i}) \,\} \\
&\quad = 1 - \sum \{\, 2^{-(i+1)} \mid i \geq 0 \wedge [x \in \bigcup \mathcal{C}^i \Rightarrow (x,y) \in \mathrm{Sym}(U_{\mathcal{C}^i})] \,\}
\end{aligned}$$

For $i \geq 0$ set

$$R_i = \{\, (x,y) \in X \times X \mid x \in \bigcup \mathcal{C}^i \Rightarrow (x,y) \in \mathrm{Sym}(U_{\mathcal{C}^i}) \,\}.$$

Then we have for $x, y \in X$ and $n \geq 0$ that

$$\begin{aligned} d_{\mathfrak{A}}(x,y) < 2^{-n} &\Leftrightarrow 1 - 2^{-n} < \sum \{\, 2^{-(i+1)} \mid (x,y) \in R_i \,\} \\ &\Leftrightarrow 0 < \sum \{\, 2^{-(i+1)} \mid (x,y) \in R_i \,\} - \sum_{i=0}^{n-1} 2^{-(i+1)} \\ &\Leftrightarrow (x,y) \in \bigcap_{i=0}^{n-1} R_i \cap \bigcup_{i \geq n} R_i. \end{aligned}$$

Let $\mathcal{R}_{\mathfrak{A}}$ be the transitive quasi-uniformity on X with subbase $\{\, R_i \mid i \in \mathbb{N} \,\}$. Since

$$\bigcap_{i=0}^{n+1} R_i \subseteq \{\, (x,y) \in X \times X \mid d_{\mathfrak{A}}(x,y) < 2^{-(n+1)} \,\} \subseteq R_n \subseteq U_{\mathcal{C}^n},$$

we obtain that

Lemma 4.5. $\mathcal{U}_{\mathfrak{A}} \subseteq \mathcal{R}_{\mathfrak{A}} = \mathcal{U}_{d_{\mathfrak{A}}}$.

It follows that $\rho = \tau(\mathcal{U}_{\mathfrak{A}}) \subseteq \tau(\mathcal{R}_{\mathfrak{A}}) = \tau(d_{\mathfrak{A}})$. For the converse inclusion remember that $\mathrm{Sym}(U_{\mathcal{C}^i})$ is an equivalence relation on $\bigcup \mathcal{C}^i$, for each $i \in \mathbb{N}$. Let D^i_j with $j \in J^i$ be the corresponding equivalence classes. Then

$$R_i = \bigcup_{j \in J^i} D^i_j \times D^i_j \cup (X \setminus \bigcup \mathcal{C}^i) \times X.$$

Lemma 4.6. *$\tau(\mathcal{R}_{\mathfrak{A}}) \subseteq \tau(\mathcal{U}_{\mathfrak{A}})$, exactly if $D^i_j \in \tau(\mathcal{U}_{\mathfrak{A}})$, for all $i \in \mathbb{N}$ and $j \in J^i$.*

Note that $D^i_j = \bigcap \mathcal{C}^i_x \setminus \bigcup(\mathcal{C}^i \setminus \mathcal{C}^i_x)$, for each $x \in D^i_j$. Here, $\mathcal{C}^i_x$ is the collection of those $C \in \mathcal{C}^i$ that contain x. Since $\bigcup_{i \in \mathbb{N}} \mathcal{C}^i$ is a subbasis for ρ, we therefore obtain that $\tau(\mathcal{R}_{\mathfrak{A}}) \subseteq \rho$ if and only if for each $i \in \mathbb{N}$ and $x \in \bigcup \mathcal{C}^i$ there are indices $j_1, \ldots, j_n \in \mathbb{N}$ and sets $C_1 \in \mathcal{C}^{j_1}, \ldots, C_n \in \mathcal{C}^{j_n}$ so that $x \in \bigcap_{\nu=1}^{n} C_\nu \subseteq \bigcap \mathcal{C}^i_x \setminus \bigcup(\mathcal{C}^i \setminus \mathcal{C}^i_x)$.

The next result is now a consequence of what we have shown so far.

Theorem 4.7. *Let (X, ρ) be a T_0 space and $\mathfrak{A} = (\mathcal{C}^i)_{i \in \mathbb{N}}$ be a family of interior-preserving collections of open subsets of X such that $\bigcup_{i \in \mathbb{N}} \mathcal{C}^i$ is a subbase for ρ. Then the following statements hold:*

(1) *The distance function is a partial metric on X.*

(2) *The kernel* $\ker(p_{\mathfrak{A}})$ *of* $p_{\mathfrak{A}}$ *is a* G_δ*-set.*
(3) *The partial metric* $p_{\mathfrak{A}}$ *is compatible with* ρ *if and only if*

$$\bigcap \mathcal{C}^i_x \setminus \bigcup (\mathcal{C}^i \setminus \mathcal{C}^i_x) \in \rho,$$

for all $i \in \mathbb{N}$ *and* $x \in \bigcup \mathcal{C}^i$*.*
(4) *With respect to the induced topology* $\ker(p_{\mathfrak{A}})$ *is a zero-dimensional space.*

Proof. It remains to show statement (4). By the preceding two lemmas the transitive quasi-uniformity $\mathcal{R}_{\mathfrak{A}}$ is compatible with ρ. Thus, the uniformity generated by the restrictions of the entourages R_i to the kernel of $p_{\mathfrak{A}}$ is a transitive as well and compatible with the induced topology $\rho{\restriction}\ker(p_{\mathfrak{A}})$. With [7, Proposition 6.1] we obtain that the subspace $(\ker(p_{\mathfrak{A}}), \rho{\restriction}\ker(p_{\mathfrak{A}}))$ is zero-dimensional. □

Obviously, the condition in Theorem 4.7(3) is satisfied if the members of each collection $\mathcal{C}^i$ are pairwise disjoint.

Corollary 4.8. *Let* (X, ρ) *be a* T_0 *space and* $\mathfrak{A} = (\mathcal{C}^i)_{i\in\mathbb{N}}$ *be a family of collections of pairwise disjoint open subsets of* X *such that* $\bigcup_{i\in\mathbb{N}} \mathcal{C}^i$ *is a subbase for* ρ*. Then the following statements hold:*
(1) *The distance function* $p_{\mathfrak{A}}$ *is a partial metric on* X *that is compatible with* ρ*.*
(2) *The kernel* $\ker(p_{\mathfrak{A}})$ *of* $p_{\mathfrak{A}}$ *is a* G_δ*-set.*
(3) *With respect to the induced topology* $\ker(p_{\mathfrak{A}})$ *is a zero-dimensional space.*

Note that in this case the partial metric $p_{\mathfrak{A}}$ coincides with the partial metric defined in [13, Proposition 2].

If the members of all $\mathcal{C}^i$ are pairwise disjoint we have that

$$U_{\mathcal{C}^i} = \bigcup_{C\in\mathcal{C}^i} C \times C \cup (X \setminus \bigcup \mathcal{C}^i) \times X,$$

which implies that $U_{\mathcal{C}^i} = R_i$. Thus the quasi-uniformities $\mathcal{U}_{\mathfrak{A}}$, $\mathcal{R}_{\mathfrak{A}}$ and $\mathcal{U}_{d_{\mathfrak{A}}}$ all coincide.

Proposition 4.9. *Let* (X, ρ) *be a* T_0 *space and* $\mathfrak{A} = (\mathcal{C}^i)_{i\in\mathbb{N}}$ *be a family of collections of pairwise disjoint open subsets of* X *such that* $\bigcup_{i\in I} \mathcal{C}^i$ *is a subbase for* ρ*. Then* $\mathcal{U}_{p_{\mathfrak{A}}} = \mathcal{R}_{\mathfrak{A}} = \mathcal{U}_{\mathfrak{A}}$*.*

Let D be an ω-algebraic domain and $\mathfrak{P} = (\mathcal{P}^z)_{z\in K_D}$ as well as $\mathfrak{S} = (\mathcal{S}^z)_{z\in K_D}$ be the families of collections of pairwise disjoint

Scott open sets considered in Example 4.2. Then we have for the partial metrics p and q defined in (3.1) and (3.2), respectively, that $p = p_{\mathfrak{P}}$ and $q = p_{\mathfrak{S}}$. Moreover, as a consequence of Corollary 4.8 we obtain that both are compatible with the Scott topology, as stated in Propositions 3.3 and 3.4.

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