On Some Constructions in Quantitative Domain Theory (Extended Abstract)

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1 Introduction

Domains introduced by Dana Scott [11] and independently by Yuri L. Ershov [3] are a structure modelling the notion of approximation and of computation. A computation performed using an algorithm proceeds in discrete steps. After each step there is more information available about the result of the computation. In this way the result obtained after each step can be seen as an approximation of the finite result.

Unlike in analytical mathematics, where natural metrics are at hand to measure the grade of an approximation, the theory of approximation based on domains was mainly of a qualitative nature. The situation started to change when M. B. Smyth [12] discovered that there is a notion of distance in domains, but it is necessarily not symmetric. Similarly, S. Matthews [8, 9] found that canonical metrics defined for the maximal elements of certain domains can be extended to the whole domain by allowing that points may have a positive self-distance, which is considered as the weight of that point. He also showed that there is a close connection between a subclass of the quasi metrics used by Smyth and his partial metrics: each partial metric defines a weighted quasi metric and vice versa. In subsequent studies [10, 13, 15] weights turned out to be a powerful tool for the introduction of partial metrics. A special class of weights are the measurements introduced by K. Martin in his thesis [7]. They are strongly intertwined with the topological structure of a domain.

An obvious question raised independently by R. Heckmann [6] and S. O'Neill [10] is which domains are partial metrizable, i.e., on which domains exists a partial metric such that its topology coincides with the Scott topology of the domain. In [10] O'Neill showed that prime-algebraic Scott domains are partial metrizable. This result has recently been extended to the class of ω-continuous domains, independently by M. Schellekens [13] and P. Waszkiewicz [15].

It follows, of course, that also the product and, if it is ω-continuous again, the space of all Scott continuous functions between such domains is partial metrizable, but if one constructs a partial metric on these domains by applying the definitions given in the above mentioned proofs one will not make use of the partial metrics coming with the components.

In this paper we study three important domain constructions, Cartesian products, function spaces and inverse limits of ω-chains of domains with embedding/projection pairs as connecting morphisms, and show how a quasi metric and a measurement, respectively, for the composed spaces can be obtained from the corresponding maps coming with the components. The domains we consider are continuous directed-complete partial orders. In the case of the function space construction we als require the range space to be bounded-complete.

For quasi-metrics our constructions resemble two well-known definitions from analytical mathematics: the sup metric and the ℓ_1 metric. In the measurement case only the latter approach seems to work. It remains the question in which way partial metrics can be transferred under the domain constructions considered here. Unfortunately, our approaches seem not to work in that case. Distance functions defined as in the quasi-metric case have nice properties, but it is not clear whether they are partial metrics again.

As has been demonstrated by Waszkiewicz [15] every measurement on a continuous poset which induces the Scott topology everywhere on the domain and satisfies a weak modularity law induces a partial metric on the poset in a natural way such that the partial metric topology coincides with the Scott topology. Note that the modularity condition generalizes a requirement used by O'Neill [10]. After having not been able to construct partial metrics on the composed domains directly from those coming with the components, this result motivated us to consider measurements instead. We show that the two properties used by Waszkiewicz do transfer under our constructions, thus giving rise to a canonical partial metric on the more complex domains.

2 Definitions and results

2.1 Quasi-metrics

Let $(D, \sqsubseteq)$ be a partial order with smallest element $\perp$. A subset S of D is called *consistent* if it has an upper bound. S is *directed*, if it is nonempty and every pair of elements in S has an upper bound in S. D is a *directed-complete* partial order (cpo) if every directed subset S of D has a least upper bound $\bigsqcup S$ in D, and D is *bounded-complete* if every consistent subset has a least upper bound in D. In a bounded-complete cpo any consistent pair $\{x, y\}$ has a least upper bound, written $x \sqcup y$. Moreover, all pairs $\{x, y\}$ have a greatest lower bound, written $x \sqcap y$. Standard references for domain theory and its applications are [5, 4, 1, 14, 2].

If $(D, \sqsubseteq)$ is a cpo and x, $y \in D$ then one says that x *approximates* y, and writes $x \ll y$ if for every directed subset S of D with $y \sqsubseteq \bigsqcup S$ there is some $u \in S$ such that $x \sqsubseteq u$. The relation $\ll$ is transitive. It is also called *way-below relation*.

Definition 2.1 Let $(D, \sqsubseteq)$ be cpo.

1. A subset Z of D is a *basis* of D if for any $x \in D$ the set $Z_x = \{\, z \in Z \mid z \ll x \,\}$ is directed and $x = \bigsqcup Z_x$.
2. D is called *continuous* if it has a basis.

As is well-known, on each cpo there is a canonical topology σ: the *Scott* topology. A subset X is open, if it is upwards closed with respect to $\sqsubseteq$ and intersects each directed subset of D of which it contains the least upper bound. In case that D is continuous with basis Z, this topology is generated by the sets $\Uparrow\{z\}(= \{\, y \in D \mid z \ll y \,\})$ with $z \in Z$.

The *product* $D \times E$ of two cpo's D and E is the Cartesian product of the underlying sets ordered coordinatewise. If both D and E are continuous, the same holds for $D \times E$.

Definition 2.2 Let D and E be cpo's. A map $f : D \to E$ is said to be *Scott continuous* if it is monotone and for any directed subset S of D,

$$f(\bigsqcup S) = \bigsqcup f(S).$$

It is well-known that Scott continuity coincides with topological continuity. Denote the collection of all continuous maps from D to E by $[D \to E]$. Endowed with the *pointwise order*, that is, $f \sqsubseteq_p g$ if $f(x) \sqsubseteq g(x)$ for all $x \in D$, it is a cpo again. If D and E are continuous, $[D \to E]$ is also continuous.

Definition 2.3 Let D and E be cpo's. A pair (e, p) of maps $e \in [D \to E]$ and $p \in [E \to D]$ is called an *embedding/projection pair* if the following two conditions hold:

- $p \circ e = \mathrm{id}_D$, the identity on D
- $e \circ p \sqsubseteq_p \mathrm{id}_E$.

The map e is called *embedding* and p *projection*.

By an ω-*chain* of continuous cpo's we understand a diagram of the form $\Delta = D_0 \overset{p_0}{\leftarrow} D_1 \overset{p_1}{\leftarrow} \cdots$, where the D_i are continuous cpo's and the maps p_i are projections. The *inverse limit* of Δ is the set D^∞ of all infinite sequences $\bar{x} \in \Pi_{i \in \omega} D_i$ such that $x_i = p_i(x_{i+1})$, for all $i \in \omega$. Endowed with the componentwise partial order D^∞ is a cpo again, which is continuous if all the D_i are continuous.

Definition 2.4 Let X be a set. A map $d \colon X \times X \to [0, \infty)$ is called a *quasi metric* if the following three conditions hold for all $x, y, z \in X$:

- $d(x, x) = 0$
- $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$
- $d(x, z) \leq d(x, y) + d(y, z)$.

Every quasi metric d defines a canonical T_0 topology τ_d on X, which is generated by the sets

$$B^d_\varepsilon(x) = \{\, y \in X \mid d(x, y) < \varepsilon \,\}$$

with $x \in X$ and $\varepsilon > 0$.

If D is a cpo and d is a quasi metric on D we say that d is *appropriate* for D, if $\tau_d = \sigma_D$.

In what follows we will always assume that the range of a quasi metric is bounded by 1: if necessary use $d(x, y)/(1 + d(x, y))$ instead of $d(x, y)$. This transformation does not change the topology of d.

Now, for $i = 1, 2$, let (D_i, d_i) be continuous cpo's with appropriate quasi metrics. We will investigate how an appropriate quasi metric can be defined on $D_1 \times D_2$ from d_1 and d_2, and similarly for the other constructions introduced above.

Theorem 2.5 *Let D_1 and D_2 be continuous cpo's with quasi metrics d_1 and d_2, respectively. For $(x, y), (x', y') \in D_1 \times D_2$ define*

$$\check{d}((x, y), (x', y')) = \max\{d_1(x, x'), d_2(y, y')\}$$

and

$$\hat{d}((x, y), (x', y')) = d_1(x, x') + d_2(y, y').$$

Then $\check{d}$ and $\hat{d}$ are both quasi metrics on $D_1 \times D_2$ that are appropriate, if d_1 and d_2 are.

Theorem 2.6 *Let D_1 be a continuous cpo with a countable basis Z and D_2 be a bounded-complete continuous cpo with a quasi metric d_2. For $f, g \in [D_1 \to D_2]$ define $\tilde{d}(f,g)$ and $\bar{d}(f,g)$, respectively in the following way:*

If Z is finite, say $Z = \{u_0, \ldots, u_n\}$, set

$$\tilde{d}(f,g) = \max\{\, d_2(f(u), g(u)) \mid u \in Z \,\}$$

and

$$\bar{d}(f,g) = \sum_{i=0}^{n} d_2(f(u_i), g(u_i)).$$

If Z is infinite, say $Z = \{u_0, u_1, \ldots\}$, set

$$\tilde{d}(f,g) = \sup\{\, 2^{-(i+1)} \cdot d_2(f(u_i), g(u_i)) \mid i \geq 0 \,\}$$

and

$$\bar{d}(f,g) = \sum_{i=0}^{\infty} 2^{-(i+1)} \cdot d_2(f(u_i), g(u_i)).$$

Then $\tilde{d}$ and $\bar{d}$ are both quasi metrics on $[D_1 \to D_2]$, which are appropriate if d_2 is.

Theorem 2.7 *Let $(D_i, p_i)_{i \in \omega}$ be an ω-chain of continuous cpo's with quasi metrics d_i. For $\bar{x}, \bar{y} \in D^\infty$ set*

$$d^\infty(\bar{x}, \bar{y}) = \sum_{i=0}^{\infty} 2^{-(i+1)} \cdot d_i(x_i, y_i).$$

Then d^∞ is a quasi metric on D^∞, which is appropriate if all the d_i are.

2.2 Measurements

As follows from the definition, the notion of a quasi metric is obtained from that of a metric by giving up the requirement that only the distance from a point to itself is zero, and the symmetry requirement. In the case of partial metrics the first of these conditions is given up as well, while the second one, symmetry, is kept.

Definition 2.8 Let X be a set. A map $p \colon X \times X \to [0, \infty)$ is a *partial metric* if the following four conditions hold for all $x, y, z \in X$:

- $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$
- $p(x,x) \leq p(x,y)$
- $p(x,y) = p(y,x)$
- $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

With each partial metric a canonical T_0 topology τ_p can be associated, which is generated by the open balls

$$B^p_\varepsilon = \{\, y \in X \mid p(x,y) < p(x,x) + \varepsilon \,\},$$

with $x \in X$ and $\varepsilon > 0$.

If D is a cpo and p is a partial metric on D the canonical topology τ_p of which coincides with the Scott topology on D, we say that p is *appropriate* for D.

For partial metrics the self-distance $p(x,x)$ of a point x needs not to be zero. It is considered as the *weight* of x. As has been shown by O'Neill [10] weights satisfying a modularity condition define a partial metric with the given weight as self-distance in a natural way. This result has been improved by Waszkiewicz [15] for a special class of weights, called measurements, which have been studied by Martin in his thesis [7] and which appear quite naturally in this context.

Let D be a cpo and let $[0,\infty)^{\mathrm{op}}$ denote the set of the nonnegative reals endowed with the converse of the natural order on the reals. Note that $[0,\infty)^{\mathrm{op}}$ is a continuos cpo. For a monotone map $\mu\colon D \to [0,\infty)^{\mathrm{op}}$ and any $x \in D$, $\varepsilon > 0$ set

$$\mu_\varepsilon(x) = \{\, y \in D \mid y \sqsubseteq x \wedge \mu(y) < \varepsilon \,\}.$$

The mapping μ is said to *induce the Scott topology* on a subset X of D, if for all $U \in \sigma$ and all $x \in X$ with $x \in U$ there is some $\varepsilon > 0$ such that $x \in \mu_\varepsilon(x) \subseteq U$. Moreover, μ is called *weakly semimodular* if for all consistent pairs $x, y \in D$ and all upper bounds u of x and y there exists a lower bound v of x and y such that

$$\mu(u) + \mu(v) \le \mu(x) + \mu(y).$$

Definition 2.9 Let D be a continuous cpo. A map $\mu\colon D \to [0,\infty)^{\mathrm{op}}$ is a *measurement* if it is Scott continuous and induces the Scott topology on its kernel $\{\, x \in D \mid \mu(x) = 0 \,\}$.

For a measurement $\mu\colon D \to [0,\infty)^{\mathrm{op}}$ the map $d_\mu\colon D \times D \to [0,\infty)$ defined by

$$d_\mu(x,y) = \inf \{\, \mu(z) \mid z \ll x, y \,\}$$

is the *distance function associated* with μ.

Proposition 2.10 (Waszkiewicz) *Let D be a continuous cpo with a weakly semimodular measurement $\mu\colon D \to [0,\infty)^{\mathrm{op}}$ that induces the Scott topology on D. Then the distance function associated with μ is a partial metric which is appropriate for D.*

For the constructions considered in the preceding section we shall now see how measurements on the composed cpo's can be defined from the measurements coming with the components. Note here that the range of a measurement μ is always contained in the real interval $[0, \mu(\perp)]$.

Theorem 2.11 *Let D_1 and D_2 be continuous cpo's with measurements μ_1 and μ_2, respectively. For $(x,y) \in D_1 \times D_2$ define*

$$\check{\mu}(x,y) = \max\{\mu_1(x), \mu_2(y)\}$$

and

$$\hat{\mu}(x,y) = \mu_1(x) + \mu_2(y).$$

Then $\check{\mu}$ and $\hat{\mu}$ are both measurements on $D_1 \times D_2$ such that the following two statements hold:

1. *If μ_1 and μ_2, respectively, induce the Scott topology on D_1 and D_2, then $\check{\mu}$ and $\hat{\mu}$ both induce the Scott topology on $D_1 \times D_2$.*

2. *If μ_1 and μ_2 are weakly semimodular, the same is true for both $\check{\mu}$ and $\hat{\mu}$.*

Theorem 2.12 *Let D_1 be a continuous cpo with a countable basis Z and D_2 be a bounded-complete continuous cpo with a measurement μ_2. For $f \in [D_1 \to D_2]$ define $\bar{\mu}(f)$ in the following way:*

If Z is finite, say $Z = \{u_0, \ldots, u_n\}$, set

$$\bar{\mu}(f) = \sum_{i=0}^{n} \mu_2(f(u_i))$$

and if Z is infinite, say $Z = \{u_0, u_1, \ldots\}$, set

$$\bar{\mu}(f) = \sum_{i=0}^{\infty} 2^{-(i+1)} \cdot \mu_2(f(u_i)).$$

Then $\bar{\mu}$ is a measurement on $[D_1 \to D_2]$ such the following two statements hold:

1. *If μ_2 induces the Scott topology on D_2, then $\bar{\mu}$ induces the Scott topology on $[D_1 \to D_2]$.*

2. *If μ_2 is weakly semimodular then $\bar{\mu}$ is as well.*

Theorem 2.13 *Let $(D_i, p_i)_{i \in \omega}$ be an ω-chain of continuous cpo's with measurements μ_i. For $\bar{x} \in D^\infty$ define*

$$\mu^\infty(\bar{x}) = \sum_{i=0}^{\infty} 2^{-(i+1)} \cdot \mu_i(x_i).$$

Then μ^∞ is a measurement on D^∞ such that the following two statements hold:

1. *If for $i \in \omega$, μ_i induces the Scott topology on D_i, then μ^∞ induces the Scott topology on D^∞.*

2. *If for $i \in \omega$, μ_i is weakly semimodular, the same holds for μ^∞.*

Note that the above results for product and function spaces can easily be extended to dependent sums and products.

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