A Refined Model Construction for the Polymorphic Lambda Calculus

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Polymorphic Lambda Calculus

(Girard, 1971; Reynolds, 1974)

Definition

The *types* of the polymorphic lambda calculus are those that can be generated by the following clauses:

1. The *type variables* $\alpha, \alpha_0, \alpha_1$ etc. are types.
2. If σ and τ are types, then $\sigma \to \tau$ is a type.
3. If σ is a type and γ is a type variable, then $\Pi\gamma.\sigma$ is a type.

Definition

The concept of a *term of type* σ, where σ is a type, is inductively defined by the following clauses:

1. For any type σ, the *variables of type* σ, $x^\sigma, x_0^\sigma, x_1^\sigma$ etc. are terms of type σ.
2. If t is a term of type τ and y^σ is a variable of type σ, then $\lambda y^\sigma.t$ is a term of type $\sigma \to \tau$.
3. If t and u are terms of respective types $\sigma \to \tau$ and σ, then $t(u)$ is a term of type τ.
4. If t is a term of type σ and γ is a type variable that is not free in the type of any variable freely occurring in t, then $\Lambda\gamma.t$ is a term of type $\Pi\gamma.\sigma$.
5. If t is a term of type $\Pi\gamma.\sigma$ and τ is a type, then $t\{\tau\}$ is a term of type $\sigma[\tau/\gamma]$.

Semantics

Problem: Interpretation of $\Pi\gamma.\tau$.

$$[\![\Pi\gamma.\tau]\!] = \prod_{[\![\text{types}]\!]} [\![\tau]\!],$$

but $\Pi\gamma.\tau \in$ types.

Reynolds, 1984:

> *There is no set-theoretical model of the polymorphic lambda calculus.*

Solution (Girard, 1986):

Let **DOM** be a category of domains such that every object in **DOM** is the colimit of an ω-chain of finite domains with embedding-projections as bonding maps.

Let τ be a type expression with free type variables $\gamma_1, \dots, \gamma_n$. Interpret τ as an ω-continuous functor

$$[\![\tau]\!] \colon (\mathbf{DOM}^{\mathbf{ep}})^n \to \mathbf{DOM}^{\mathbf{ep}}$$

and $\Pi\gamma.\tau$ as the collection of its continuous sections.

Problem: This collection is too large to be a set.

Important observation: Every continuous section of $[\![\tau]\!]$ is uniquely determined by its behaviour on the finite domains in **DOM**.

Let $\mathbb{S}$ be a countable full subcategory of $\mathbf{DOM}^{\mathbf{ep}}$ which up to isomorphism contains every finite domain in DOM.

Set

$$[\![\Pi\gamma.\tau]\!] = \{\text{continuous section of } [\![\tau]\!] \restriction \mathbb{S}\}.$$

Note that this is a domain again with respect to the pointwise order.

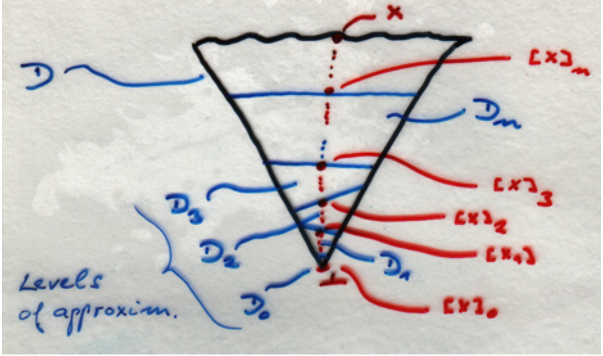
Interpretation of terms:

Let t be a term of type τ with free variables $x_1^{\sigma_1}, \ldots, x_n^{\sigma_n}$. Set

$$[\![t]\!] = \text{continuous section of } [\![\tau]\!].$$

Girard uses qualitative domains, but he does not fully exploit their approximability by finite domains.

Each such domains is a colimit of an ω-chain of finite subdomains.



- Provides a measure of how good an approximation z of x is: take the smallest n so that $z \in D_n$.
- For any $x \in D$, there is a best approximation of x with respect to each level D_n:

$$[x]_n = \bigsqcup \{ z \in D_n \mid z \sqsubseteq x \}.$$

Definition

Let $(D, \sqsubseteq)$ be a poset and $x \in D$. Then x is

1. *compact* if for all directed $S \subseteq D$ with least upper bound in D,
$$x \sqsubseteq \bigsqcup S \Rightarrow (\exists u \in S) x \sqsubseteq u.$$
2. *completely prime* if for all bounded $S \subseteq D$ with least upper bound in D,
$$x \sqsubseteq \bigsqcup S \Rightarrow (\exists u \in S) x \sqsubseteq u.$$

Set

$$D^0 = \{ x \in D \mid x \text{ is compact} \}$$
$$D^p = \{ x \in D \mid x \text{ is completely prime} \}$$

Definition

D is a *pre-dI-domain* if

- Every directed $S \subseteq D$ has a least upper bound in D
- All bounded $\{x, y\} \subseteq D$ have a least upper bound in D.
- For all $x, y, z \in D$ such that $\{y, z\}$ is bounded and $x \sqcap (y \sqcup z)$, $x \sqcap y$, $x \sqcap z$ exist in D,

$$x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z).$$

- For all $x \in D^0$, $\downarrow\{x\}$ is finite.

Definition

D is a *qualitative pre-domain* if

- D is a pre-dI-domain,
- The elements of D^p are pairwise incomparable with respect to $\sqsubseteq$.

Note. $x \in D$ is uniquely determined by $\{\, p \in D^p \mid p \sqsubseteq x \,\}$.

Definition

Let D be a qualitative pre-domain. Then $([\cdot]_i^D \colon D \to D)_{i\in\omega}$ is an *approximation structure* on D if for all $i, j \in \omega$ and $x, y \in D$,

- $[\cdot]_i^D$ is stable
- $\downarrow D_i \subseteq D_i$, where $D_i = \{\, x \in D \mid [x]_i^D = x \,\}$.
- $D^0 \subseteq \bigcup_\nu D_\nu$
- $[\cdot]_i^D \circ [\cdot]_j^D = [\cdot]_{\mathrm{min}\{i,j\}}^D$
- $[\cdot]_i^D \sqsubseteq_s \mathrm{id}_D$
- $\bigsqcup_\nu [\cdot]_\nu^D = \mathrm{id}_D$
- $[x]_0^D = [y]_0^D$.

Note.

- All conditions are universally quantified. Thus, $(\emptyset, \emptyset, (\emptyset)_{i \in \omega})$ is a qualitative pre-domain with approximation structure.
- If D is nonempty, then D is a qualitative domain with least element $[x]_0^D$.

Aim. Interpret types by qualitative pre-domains with approximation structure.

Definition

For $x \in D$ the *rank* $\mathrm{rk}(x)$ of x is given by

$$\mathrm{rk}(x) = \begin{cases} \min \{ i \mid x \in D_i \}, & \text{if } \{ i \mid x \in D_i \} \neq \emptyset \\ \omega, & \text{otherwise.} \end{cases}$$

The approximation structure is determined by the ranks of the complete primes.

Lemma

$$\mathrm{rk}(x) = \sup\{\, \mathrm{rk}(p) \mid p \in D^p, p \sqsubseteq x \,\}.$$

Note that for $p \in D^p$, $\mathrm{rk}(p) < \omega$, as $D^p \subseteq D^0 \subseteq \bigcup_\nu D_\nu$.

Lemma

Let D be a qualitative domain or empty, and $r\colon D^p \to \omega$. Set

$$[x]_i^D = \bigsqcup \{\, p \in D^p \mid p \in D^p, r(p) \leq i \,\}.$$

Then $([\cdot]_i^D)_{i\in\omega}$ is an approximation structure on D with $\mathrm{rk}(p) = r(p)$, for $p \in D^p$.

Assume. $\{\, p \in D^p \mid r(p) \leq i \,\}$ is finite, for all $i \in \omega$. Then D_i is finite as well.

Definition

Let D, E be qualitative pre-domains with approximation structure. A map $f\colon D \to E$ is *rank-preserving* if for all $x \in D$, and $i, j \in \omega$ with $j \geq i$,

$$[f(x)]_i^E = [f([x]_j^D)]_i^E.$$

Note.

- f is rank-preserving iff for all $x, y \in D$ and $i \in \omega$,

$$[x]_i^D = [y]_i^D \Rightarrow [f(x)]_i^E = [f(y)]_i^E.$$

- The empty map is rank-preserving if D is empty.

Let

$$[D \to_{srp} E] = \{\, f\colon D \to E \mid f \text{ stable, rank-preserving}\,\}.$$

Every stable map f is uniquely determined by its *trace*

$$\mathrm{tr}(f) = \{ (u,p) \in D^0 \times E^p \mid u \text{ least with } p \sqsubseteq f(u) \}.$$

Lemma

For $f \in [D \to_{srp} E]$, $\mathrm{tr}(f)$ *satisfies*

1. $(\forall (u_1, p_1), \ldots, (u_n, p_n) \in \mathrm{tr}(f))[\{u_1, \ldots, u_n\}\uparrow \Rightarrow \{p_1, \ldots, p_n\}\uparrow]$.
2. $(\forall (u,p), (u',p') \in \mathrm{tr}(f))[\{u, u'\}\uparrow \Rightarrow u = u']$.
3. $(\forall (u,p) \in \mathrm{tr}(f))\, \mathrm{rk}(u) \leq \mathrm{rk}(p)$.

Lemma

1. *From its trace f can be computed via*

$$f(x) = \bigsqcup \{ p \mid (\exists u \sqsubseteq x)(u,p) \in \mathrm{tr}(f) \}. \qquad (*)$$

2. *If* $X \subseteq D^0 \times E^p$ *with (1-3), then* X *is the trace of the stable rank-preserving map given by* $(*)$.

For $f \in [D \to_{srp} E]$ set

$$f \sqsubseteq_s g \Leftrightarrow \mathrm{tr}(f) \subseteq \mathrm{tr}(g),$$
$$[f]_i^{\to}(x) = [f(x)]_i^E.$$

Proposition

$([D \to_{srp} E], \sqsubseteq_s, ([\cdot]_i^{\to})_{i \in \omega})$ *is a qualitative pre-domain with approximation structrure.*

Note.

- $\mathrm{tr}([f]_i^{\to}) = \{ (u, p) \in \mathrm{tr}(f) \mid \mathrm{rk}(p) \leq i \}$.
- $[D \to_{srp} E]^p = \{ f \in [D \to_{srp} E] \mid \|\mathrm{tr}(f)\| = 1 \}$.
- $f \in [D \to_{srp} E]^p, \mathrm{tr}(f) = \{(u, p)\} \Rightarrow \mathrm{rk}(f) = \mathrm{rk}(p)$.

Consequently, $\{ f \in [D \to_{srp} E]^p \mid \mathrm{rk}(f) \leq i \}$ is finite.

Every qualitative domain is a colimit of an ω-chain of finite qualitative domains with embeddings as bonding maps.

Now. Embeddings must preserve the approximation structure!

Definition

Let D, E be qualitative domains with approximation structure and $e\colon D \to E]$, $p\colon E \to D$ be stable maps. Then (e, p) is a *rigid embedding/projection pair* if

- $p \circ e = \mathrm{id}_D$
- $e \circ p \sqsubseteq_s \mathrm{id}_E$.

Notation: $p = e^R$.

In addition: Embeddings must commute with the approximation maps:

$$e([x]_i^D) = [e(x)]_i^E \qquad (x \in D, i \in \omega).$$

Note.

- Subspace inclusion commutes with the approximation maps.
- $e^R([y]_i^E) = [e^R(y)]_i^D$.

Let $\mathbf{qPA^e}$ be the category of qualitative pre-domains with approximation structure and rigid embeddings that commute with the approximation maps.

Then $\emptyset$ is an isolated object: there are no arrows from/to other objects.

Proposition

Every object in $\mathbf{qPA^e}$ *is a colimit of an* ω*-chain in* $\mathbf{qPA^e}$ *of finite objects.*

The Function Space Functor

Let

$$F(D,E) = [D \to_{srp} E]$$

and for $d \in \mathbf{qPA^e}[D, D']$ and $e \in \mathbf{qPA^e}[E, E']$,

$$F(d,e)(h) = e \circ h \circ d^R \qquad (h \in F(D,E))$$
$$F(d,e)^R(h') = e^R \circ h' \circ d \qquad (h' \in F(D',E')).$$

Proposition

The function space functor F is stable and rank-preserving, i.e. for all $D, E \in \mathbf{qPA^e}$ and all $i, j \in \omega$ with $j \geq i$,

$$F(D_j \hookrightarrow D, E_j \hookrightarrow E) \upharpoonright F(D_j, E_j)_i \colon F(D_j, E_j)_i \xrightarrow{iso} F(D,E)_i.$$

Note. $D \mapsto D_i$ defines an approximation structure on $\mathbf{qPA^e}$.

The Product Construction

Definition

Let $G\colon \mathbf{qPA^e} \to \mathbf{qPA^e}$ be a stable functor. Then $(t(X))_{X\in\mathbf{qPA^e}}$ is a *uniform family* of G if for all $X, Y \in \mathbf{qPA^e}, f \in \mathbf{qPA^e}[X, Y])$

- $t(X) \in G(X)$,
- $t(X) = G(f)^R(t(Y))$.

Proposition

Let G be stable and rank-preserving and t be a uniform family of G. Then t is rank-preserving, i.e. for all $X \in \mathbf{qPA^e}$ and all $i, j \in \omega$ with $j \geq i$,

$$[t(X)]_i^{G(X)} = G(X_j \hookrightarrow X)([t(X_j)]_i^{G(X_j)}).$$

Set

$$\prod G = \{\, t \mid t \text{ is a uniform familiy of } G \,\}.$$

Note. $(\exists X \in \mathbf{qPA}) G(X) = \emptyset \Rightarrow \prod G = \emptyset \in \mathbf{qPA}$.

Assume: $G(X) \neq \emptyset$, for all $X \in \mathbf{qPA}$.

Theorem (Normal Form Theorem)

Let G be stable and rank-preserving, $X \in \mathbf{qPA}$, and $p \in G(X)^p$. Then there exist a finite $\widehat{X} \in \mathbf{qPA}$, $f \in \mathbf{qPA^e}[\widehat{X}, X]$ and $\hat{p} \in G(\widehat{X})^p$ such that

- $p = G(f)(\hat{p})$ *(normal form of p with respect to $G(\widehat{X})$)*
- $\mathrm{rk}(\widehat{X}) \leq \mathrm{rk}(\hat{p})$
- *For all $Y \in \mathbf{qPA}, f' \in \mathbf{qPA^e}[Y, X], y \in G(Y)^p$ with $p = G(f')(y)$ there is exactly one $h \in \mathbf{qPA^e}[\widehat{X}, Y]$ so that*

$$y = G(h)(\hat{p}) \quad \text{and} \quad f = f' \circ h.$$

As in (Girard, 1986): $\prod G$ is a qualitative domain.

For $t \in \prod G$ and $i \in \omega$ set

$$[t]_i^{\prod G}(X) = [t(X)]_i^{G(X)}.$$

Lemma

$([\cdot]_i^{\prod G})_{i\in\omega}$ is an approximation structure on $\prod G$.

Let $G \colon (\mathbf{qPA^e})^{m+1} \to \mathbf{qPA^e}$ be stable and rank-preserving.
For $Y \in \mathbf{qPA^e}$ and $f \in \mathbf{qPA^e}[Y, Y']$ set

$$G_{\vec{X}}(Y) = G(\vec{X}, Y)$$
$$G_{\vec{X}}(f) = G(\mathrm{id}_{\vec{X}}, f).$$

Then $\prod^G \colon (\mathbf{qPA^e})^m \to \mathbf{qPA^e}$ with

$$\prod^G(\vec{X}) = \prod G_{\vec{X}}$$

can be made into a stable rank-preserving functor.

Model:

type expression	stable rank-preserving functor
term	uniform family

Advantages:

- Absurdity $\Pi\alpha.\alpha$ is interpreted by $\emptyset$ (as it should be!).
- The interpretation of arrow types is smaller as in Girard's model.
- The approximability of domains by finite domains is fully taken into consideration.

However

- The interpretation of

$$\text{Polybool} = \Pi\alpha.\alpha \to (\alpha \to \alpha)$$

still consists of

$$\textsc{True}, \quad \textsc{False}, \quad \textsc{Inter},$$

where

$$\textsc{Inter} = \Lambda X.\lambda x.\lambda y.x \sqcap y.$$

Solution. Restrict to *total* domain elements.

Girard: No requirements: Any $D' \subseteq D$ is a set of total elements.

Obviously, this definition is much too general. Intuitive requirements for an element to be total are that it is

- completely specified,
- the result of an infinite approximation process .

Here, we will require that it has at least infinite rank.

Definition

Let D be a qualitative pre-domain with approximation structure. $D^t \subseteq D$ is a *totality* on D, if $\mathrm{rk}(x) = \omega$, for all $x \in D^t$, in case that $\mathrm{rk}(D) = \omega$.

Obviously, if $\mathrm{rk}(D) < \omega$ then $D^t = \emptyset$.

Let $\mathbf{qPAT^e}$ be the full subcategory of $\mathbf{qPA^e}$ of qualitative pre-domains with approximation structure and totality.

Lemma

Let $(D, D^t), (E, E^t)$ be qualitative pre-domains with approximation structure and totality and set

$$[D \to_{srp} E]^t = \{\, f \in [D \to_{srp} E] \mid f(D^t) \subseteq E^t \,\}.$$

Then $[D \to_{srp} E]^t$ is a totality on $[D \to_{srp} E]$.

Lemma

Let $G\colon \mathbf{qPAT^e} \to \mathbf{qPAT^e}$ be a stable rank-preserving functor and set

$$(\prod G)^t = \{\, t \in \prod G \mid (\forall X \in \mathbf{qPAT^e})[\mathrm{rk}(X) = \omega \Rightarrow t(X) \in G(X)^t] \,\}$$

Then $(\prod G)^t$ is a totality on $\prod G$.

In the modified model the only total elements of POLYBOOL are TRUE and FALSE. Similarly for POLYNAT.