Steering and Entropic Uncertainty Relations

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September 5, 2017

Summary

1. Steering
   - LHS model
   - Entropic steering criteria: Shannon entropy
   - Entropic steering criteria: Tsallis entropy

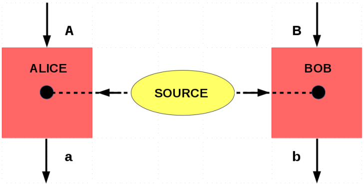
2. Applications
   - Werner states
   - Isotropic states
   - General two-qubit states
   - One-way steerable states

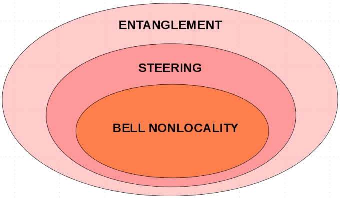
3. Final remarks

Local hidden state model

- Assume that Alice and Bob share a quantum state ρ_{AB}.
- Alice makes measurements on her system and claims that with these measurements she can steer the state inside Bob's laboratory.
- Local hidden state model:

$$p(a, b|A, B) = \sum_\lambda p(\lambda) p(a|A, \lambda) \mathrm{Tr}_B[B_b \rho_\lambda^B].$$





Entropic steering criteria

- Shannon entropy: $S(\mathcal{A}) = -\sum_k a_k \log(a_k)$,

- Relative entropy: $D(\mathcal{A}||\mathcal{B}) = \sum_k a_k \log\left(\frac{a_k}{b_k}\right)$

- If $\mathcal{A}_1, \mathcal{A}_2$ are independent distributions, with $\mathcal{A}(x,y) = \mathcal{A}_1(x)\mathcal{A}_2(y)$ - and similar for $\mathcal{B}_1, \mathcal{B}_2$,

$$D(\mathcal{A}||\mathcal{B}) = D(\mathcal{A}_1||\mathcal{B}_1) + D(\mathcal{A}_2||\mathcal{B}_2).$$

Entropic steering criteria

- Consider the quantity: $F(A,B) = -D(A \otimes B \| A \otimes \mathbb{1})$[1].

$$F(A,B) = -\sum_{i,j} p_{ij} \log\left(\frac{p_{ij}}{p_i 1/N}\right) = S(A,B) - S(A) - \log(N).$$

- Now, for a product distribution, $p(a,b|A,B) = \sum_\lambda p(\lambda) p(a|A,\lambda) \mathrm{Tr}_B[B_b \rho_\lambda^B]$,

$$\begin{aligned} F(A,B) &\geqslant -\sum_\lambda p(\lambda)\Big[D(p(a|A,\lambda)\|p(a|A,\lambda)) + D(p_q(b|B,\lambda)\|1/N)\Big] \\ &= \sum_\lambda p(\lambda) S(B|\lambda) - \log(N). \end{aligned}$$

[1] AB = joint measurement distribution ($p_{ij} = \mathrm{Tr}[(A_i \otimes B_j)\rho]$),
A = marginal distribution ($p_i = \mathrm{Tr}[(A_i \otimes \mathbb{1})\rho]$),
$\mathbb{1}$ = equal distribution: $q_j = 1/N\, \forall j$.

Entropic steering criteria

- Consider a set of measurements $A_k \otimes B_k$:

$$\sum_k \Big[S(B_k|A_k) - \log(N)\Big] \geqslant \sum_k \left(\sum_\lambda p(\lambda) S(B_k|\lambda) - \log(N)\right).$$

- If B_k obey some entropic uncertainty relation $\sum_k S(B_k) \geqslant C_B^S$, we have for all unsteerable states:

$$\sum_k \Big[S(B_k|A_k)\Big] \geqslant C_B^S.$$

- $k = 2$:

$$S(B_1) + S(B_2) \geqslant \log(\Omega), \qquad \Omega \equiv \min_{i,j}\left(\frac{1}{|\langle B_1^i|B_2^j\rangle|^2}\right).$$

- The same criteria obtained by Walborn et. al. [PRL **106**, 130402 (2011)].

Entropic steering criteria: Tsallis entropy

- Tsallis entropy: $S_q(\mathcal{A}) = \frac{1}{q-1}\left[1 - \sum_i a_i^q\right]$.
- Tsallis relative entropy: $D_q(\mathcal{A}\|\mathcal{B}) = \frac{1}{1-q}\left[1 - \sum_i \frac{a_i^q}{b_i^{q-1}}\right]$.
- Product distributions:
  $D(\mathcal{A}\|\mathcal{B}) = D(\mathcal{A}_1\|\mathcal{B}_1) + D(\mathcal{A}_2\|\mathcal{B}_2) + (q-1)D(\mathcal{A}_1\|\mathcal{B}_1)D(\mathcal{A}_2\|\mathcal{B}_2)$

$$\sum_k \left[S_q(B_k|A_k) + (1-q)C(A_k, B_k)\right] \geqslant C_B^T.$$

- In terms of probabilities,

$$\frac{1}{q-1}\left[\sum_k \left(1 - \sum_{ij} \frac{(p_{ij}^{(k)})^q}{(p_i^{(k)})^{q-1}}\right)\right] \geqslant C_B^T.$$

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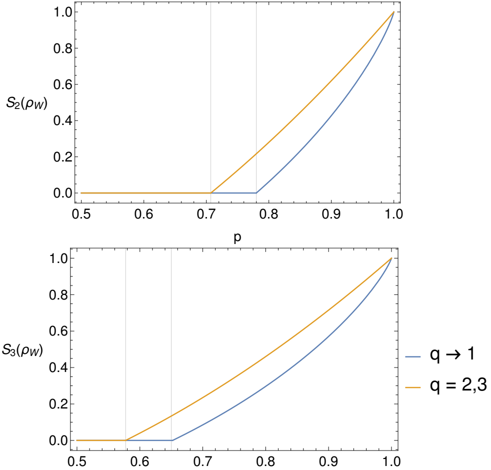
[2] $C(A,B) = \sum_i p_i^q (\ln_q(p_i))^2 - \sum_{i,j} p_{ij}^q \ln_q(p_i)\ln_q(p_{ij})$, and $\ln_q(x) \equiv \frac{x^{1-q}-1}{1-q}$ is the q-logarithm.

Application: Two-qubit Werner states

$$\rho_W = \frac{(1-p)}{4}\mathbb{1} + p|\psi\rangle\langle\psi|, \quad \text{where} \quad |\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$



Application: Isotropic states

- Isotropic states:

$$\rho_{iso} = p|\phi_d^+\rangle\langle\phi_d^+| + \frac{1-p}{d^2}\mathbb{1}, \quad \text{where} \quad |\phi^+\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|i\rangle\otimes|i\rangle.$$

- Consider a set of mutually unbiased bases in an arbitrary dimension.

- $p_i = 1/d$ for all i,
- $p_{ii} = (1+(d-1)p)/d^2$ (occurring d times),
- $p_{ij} = (1-p)/d^2$ ($i \neq j$ – occurring $d(d-1)$ times).

- These probabilities are the same for all measurements.

$$\frac{m}{q-1}\left[1 - \frac{1}{d^q}((1+(d-1)p)^q + (d-1)(1-p)^q)\right] \geqslant C_B^T,$$

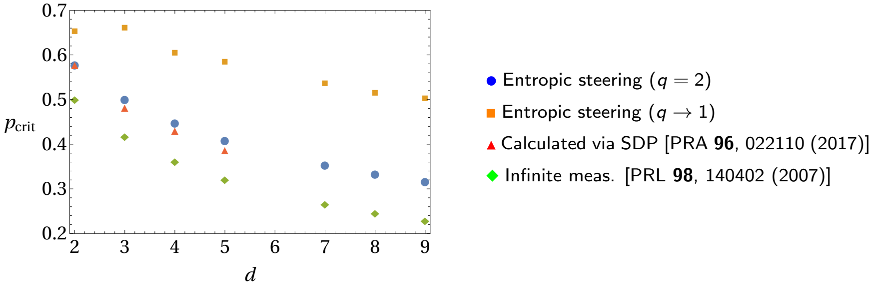
where $C_B^T = m\ln_q\left(\frac{md}{d+m-1}\right)$ for $q \in (0;2]$[3].

[3] A. E. Rastegin, Eur. Phys. J. D **67**, 269 (2013).

Application: Isotropic states

- For a complete set of MUBs (if it exists) and $q = 2$, the violation of the generalized entropic steering criteria occurs for

$$p > \frac{1}{\sqrt{d+1}}.$$



Application: General two-qubit states

$$\rho = \frac{1}{4}\left(\mathbb{1}^{AB} + \vec{a}\cdot\vec{\sigma}^{A}\otimes\mathbb{1}^{B} + \mathbb{1}^{B}\otimes\vec{b}\cdot\vec{\sigma}^{B} + \sum_{i=1}^{3} c_i\,\sigma_i^{A}\otimes\sigma_i^{B}\right)$$

- Entropic steering for three measurements ($q = 2$):

$$S_3^e(\rho) = \max\left\{0,\quad 1 - \sum_{i=1}^{3}\left[\frac{1 - a_i^2 - b_i^2 - c_i^2 + 2a_i b_i c_i}{2(1 - a_i^2)}\right]\right\}.$$

- Linear steering quantifier for three measurements ($|\sum_{k=1}^{n}\langle A_k\otimes B_k\rangle| \leqslant \sqrt{3}$, where $A_i = \hat{u}_i\cdot\vec{\sigma}$, $B_i = \hat{v}_i\cdot\vec{\sigma}$):

$$S_3^l(\rho) = \max\left\{0, \frac{\sqrt{c_1^2 + c_2^2 + c_3^2} - 1}{\sqrt{3} - 1}\right\}$$

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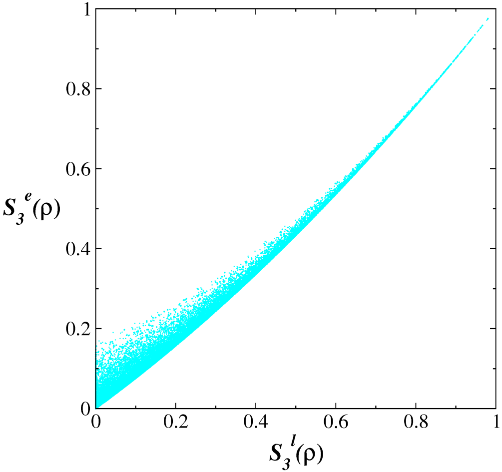
[4] A. C. S. Costa and R. M. Angelo, PRA 93, 2, 020103(R) (2016).

Application: General two-qubit states



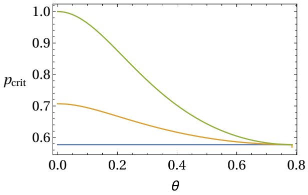
Application: One-way steerable states

$$\rho_{AB} = p|\psi(\theta)\rangle\langle\psi(\theta)| + (1-p)\mathbb{1}/2 \otimes \rho_B^\theta,$$

where $|\psi(\theta)\rangle = \cos(\theta)|00\rangle + \sin(\theta)|11\rangle$ and $\rho_B^\theta = \mathrm{Tr}_A\Big[|\psi(\theta)\rangle\langle\psi(\theta)|\Big]$.

- For three measurement settings, this state is one-way steerable[5] for $\theta \in [0, \pi/4]$ if $\frac{1}{\sqrt{3}} < p \leqslant 1/\sqrt{1+2\sin^2(2\theta)}$.
- Generalized entropic steering criteria and $q = 2$: this state is one-way steerable for

$$\frac{\sqrt{3-\sqrt{1+8\sin^2(2\theta)}}}{2\cos(2\theta)} < p \leqslant \frac{1}{\sqrt{1+2\sin^2(2\theta)}}.$$



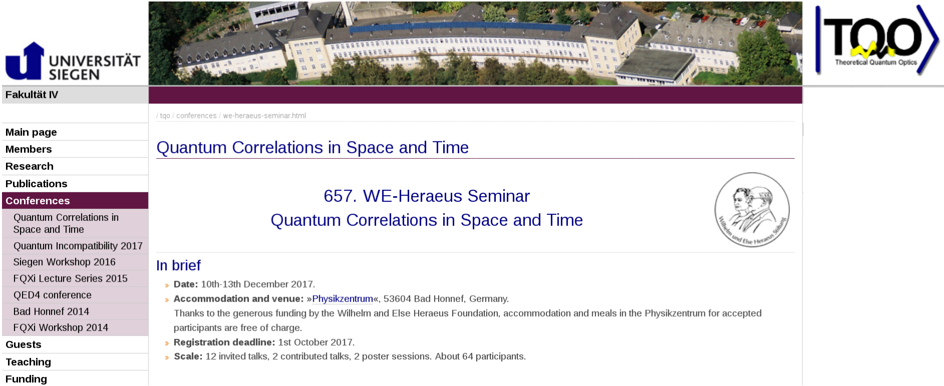
[5] Y. Xiao et. al, Phys. Rev. Lett. **118**, 140404 (2017).

Final remarks

- Generalized entropic steering criteria based on Tsallis entropy (paper in preparation).
- Applications: isotropic states, general two-qubit states, one-way steerable states.
- Future work: Extension to multipartite systems; entropic uncertainty relations in the presence of quantum memory; bound entangled states.

Thank you for your attention!

Muito obrigada!



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