Risk Measures and Counterparty Risk

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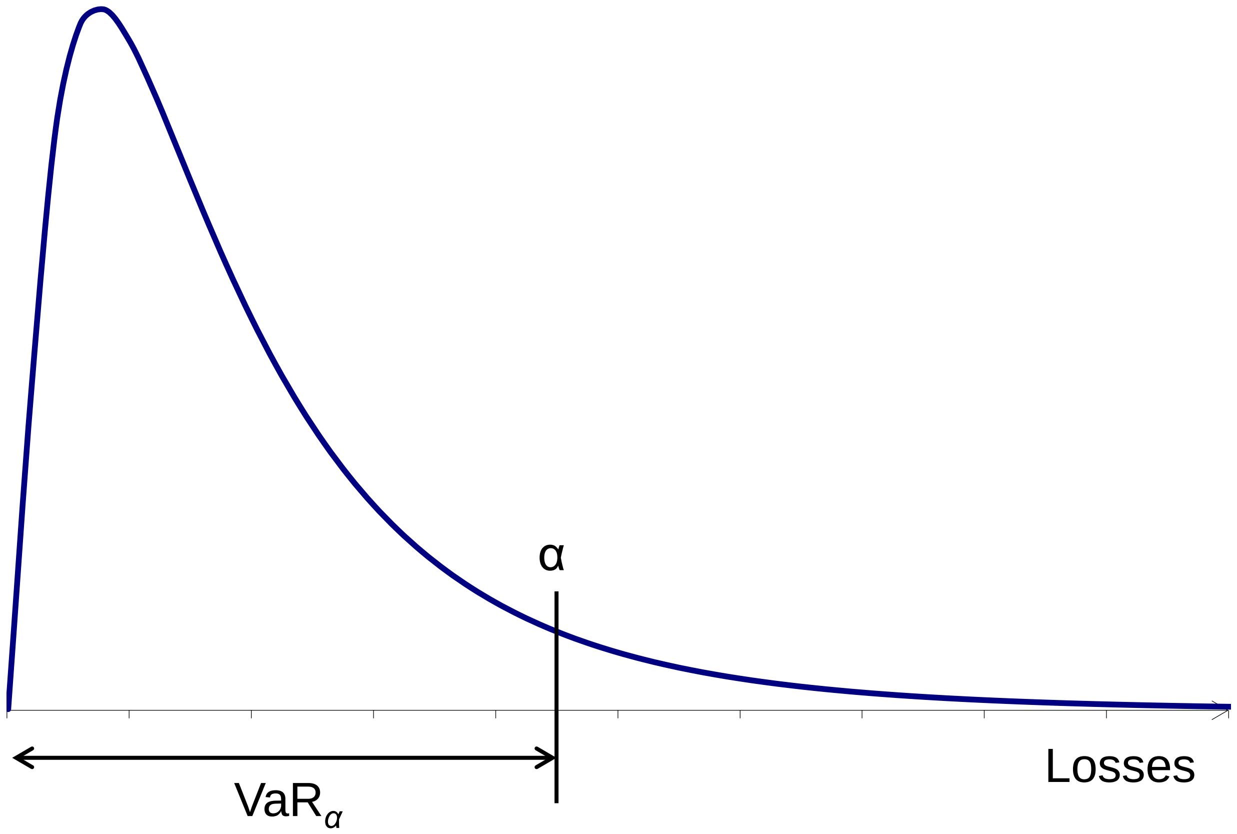
The Allocation Problem

- Total exposure normalized to 1
- $n \geq 2$ counterparts
  - known default probabilities p_i
  - known dependence structure
  - otherwise equal offers
- What is the optimal allocation?
  - What does „optimal“ mean?

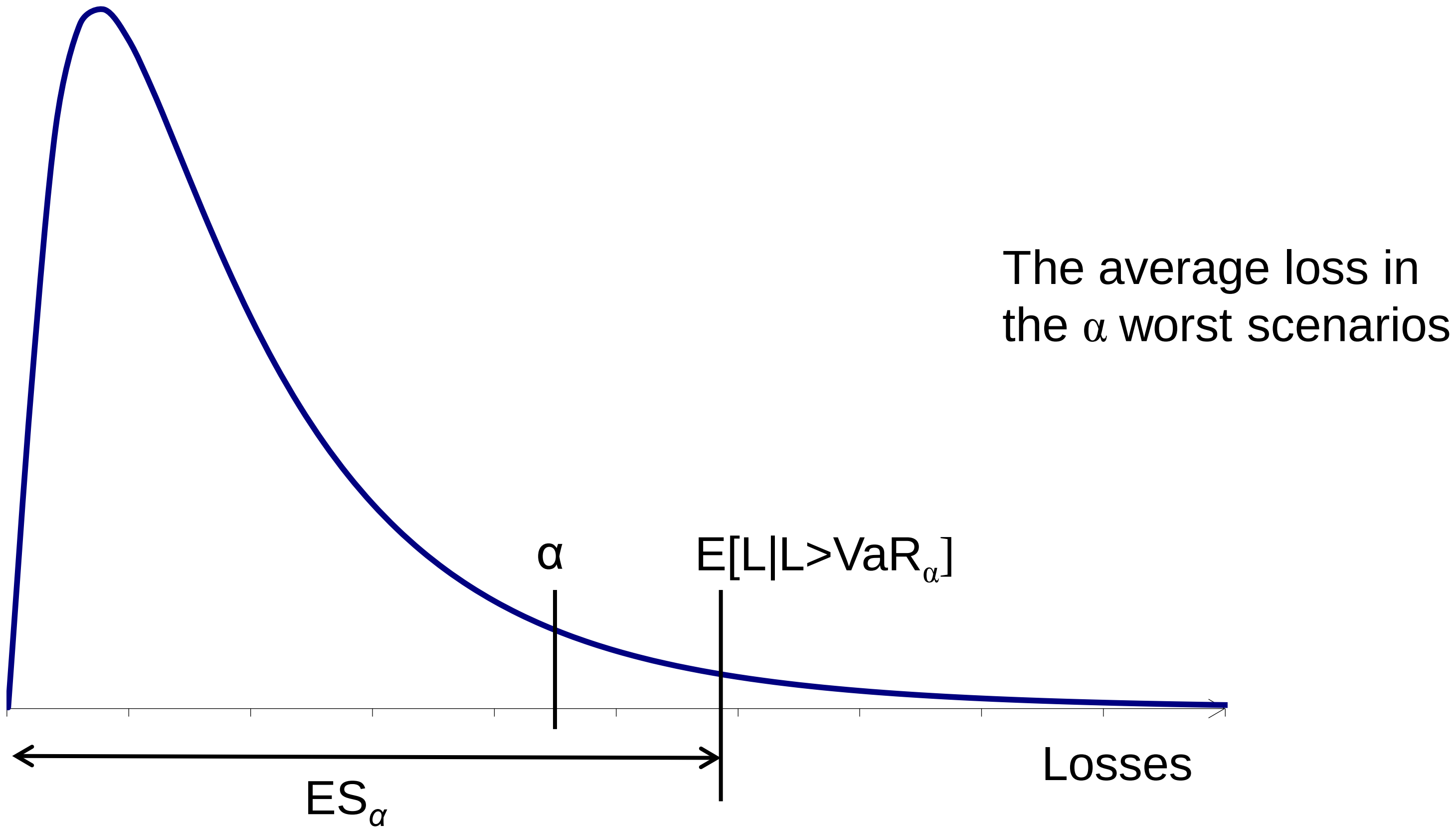
Agenda

- **Some Basics of Risk Measures**
  - Value at Risk, Expected Shortfall, Coherent and Spectral Risk Measures
- Examples and Results
  - Risk Measures as an Approach to the Allocation Problem
  - Consequences and Limitations
- Expected Utility Framework
  - Analytical Derivation of Optimal Allocations
- Lessons Learned
  - Implications for Risk Governace

Value at Risk



Expected Shortfall



Expected Shortfall as a Coherent Risk Measure

1. Monotonicity:
$$\mathrm{ES}(X) \leq \mathrm{ES}(Y) \text{ for } X \geq Y$$

2. Positive Homogeneity:
$$\mathrm{ES}(a \cdot X) = a \cdot \mathrm{ES}(X) \text{ for } a > 0$$

3. Translation Invariance:
$$\mathrm{ES}(X + b) = \mathrm{ES}(X) - b$$

4. Subadditivity:
$$\mathrm{ES}(X + Y) \leq \mathrm{ES}(X) + \mathrm{ES}(Y)$$

Spectral Risk Measures

- Subclass of coherent risk measures
- Defined by a „risk spectrum“ φ whichs weights the importance of the quantiles for the risk assessment

$$R_\varphi(X) = -\int_0^1 F_X^{-1}(q)\,\varphi(q)\,dq$$

- Expected Shortfall is a spectral risk measure with risk spectrum

$$\varphi(q) = \begin{cases} 1/\alpha, & q \le \alpha \\ 0, & q > \alpha \end{cases}$$

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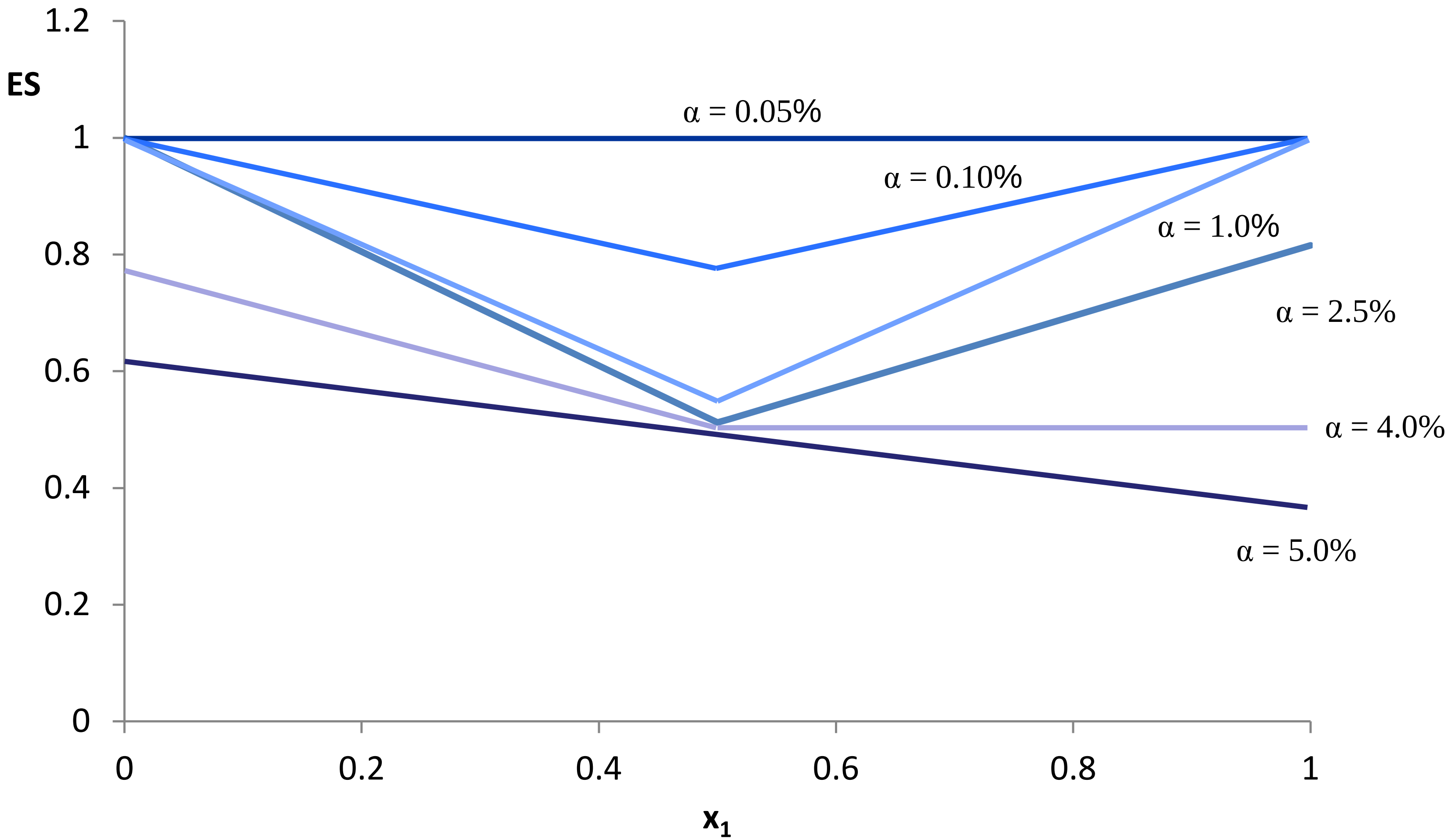
Example: 2 Counterparts

- $p_1 = 2\%;\ p_2 = 3\%;\ \rho = 0$
- Minimize Expected Shortfall (with $\alpha = 2.5\%$)!
- Allocation: $x_1 \geq x_2 = 1 - x_1$

$$L = \begin{cases} 1 & 0.06\% \\ x_1 & 1.94\% \\ 1 - x_1 & 2.94\% \\ 0 & 95.06\% \end{cases}$$

$$ES_{2.5\%} = \frac{0.06\% \cdot 1 + 1.94\% \cdot x_1 + 0.50\% \cdot (1 - x_1)}{2.5\%}$$
$$= 0.224 + 0.0144 \cdot x_1$$

Expected Shortfall for Different Quantiles



Interim Conclusion

- Expected Shortfall as a decision tool yields polar solutions: Depending on α, either $x_1 = 1$ or $x_1 = 0.5$ is optimal

- Economic Intuition
  - For $\alpha \to 100\%$, $\mathrm{ES} \to \mathrm{E}$
  - For small values of α, the avoidance of large losses is dominant

- Technical Reason
  - For a given loss distribution, $\mathrm{ES}(x_1)$ is linear
  - Only corner solutions can be optimal

Corner Solutions (n = 3)

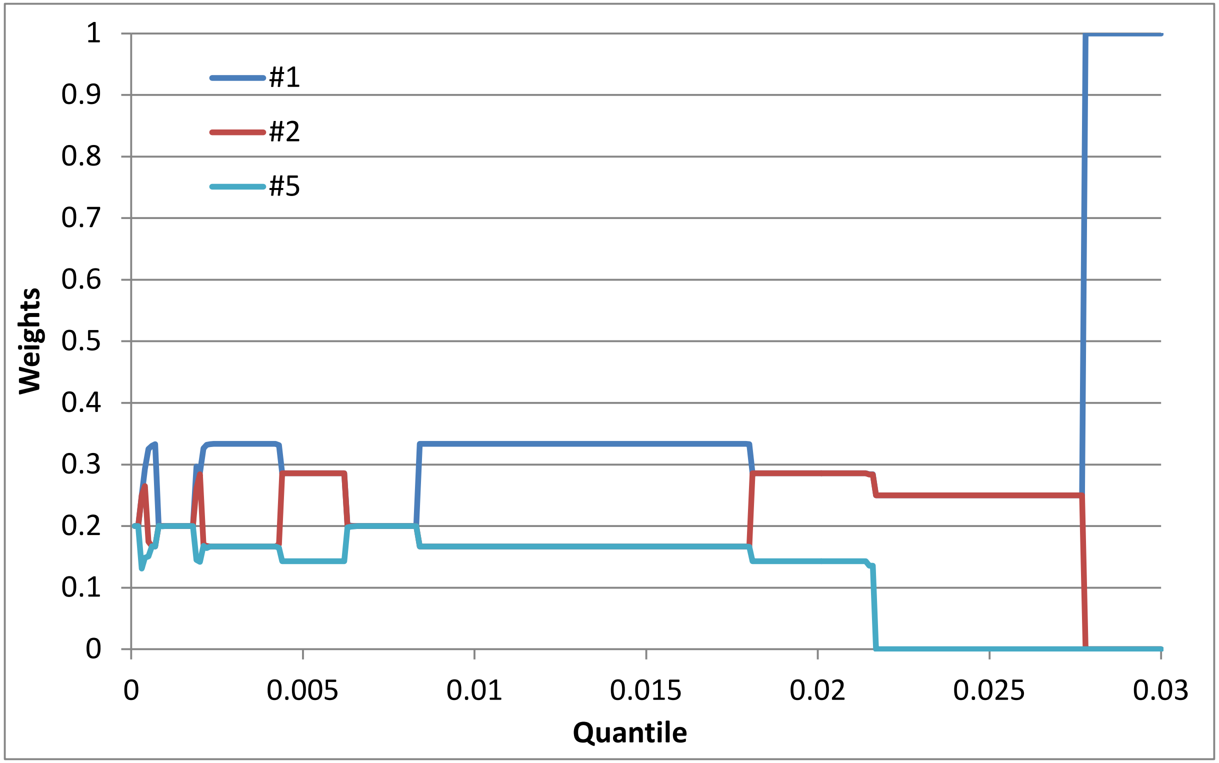
- Four possible corner solutions
  a) $x_1 = 1.00; x_2 = 0.00; x_3 = 0.00$
  b) $x_1 = 0.50; x_2 = 0.50; x_3 = 0.00$
  c) $x_1 = 0.50; x_2 = 0.25; x_3 = 0.25$
  d) $x_1 = 0.33; x_2 = 0.33; x_3 = 0.33$

- Example: $p_1 = 1.0\%$, $p_2 = 2.0\%$, $p_3 = 3.0\%$, $\rho = 0.3$

Quantile	0.2%	0.5%	1%	2%	5%	10%
Solution	d	b	d	d	a	a

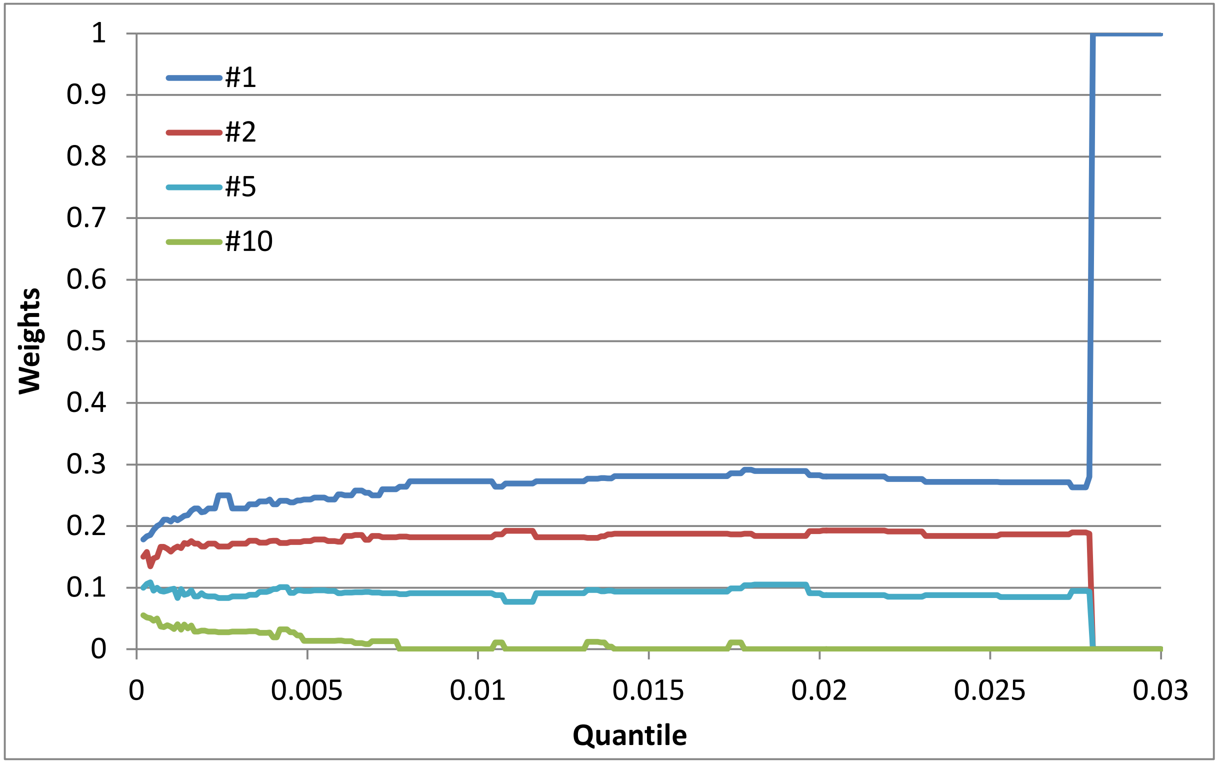
Solutions for n = 5

$p_j = j\,\%;\ \rho = 0.3$



Solutions for n = 10

$p_j = j\,\%;\ \rho = 0.3$



Risk Measures for Decision Making?

- Original scope of
  coherent risk measures (Artzner et al., 1999) and
  spectral risk measures (Acerbi, 2002):
  regulatory context (**side condition**)

- Application for economic decision making (**target function**):
  - Portfolio selection (Adam et al., 2008)
  - Credit portfolio management (Iscoe et al., 2012)

- Translation invariance and positive homogeneity invoke
  tendencies to corner solutions
  (Brandtner, 2013; Brandtner/Kürsten, 2015)

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Maximizing Expected Utility (2 Counterparts)

- $\mathrm{E}\,u(x) \rightarrow \text{Max!}$
- Current wealth level c
- Final wealth level $x = c - L$

$$x = \begin{cases} c-1 & p_{12} \\ c-x_1 & p_1 \\ c-(1-x_1) & p_2 \\ c & p_\emptyset \end{cases}$$

$$Eu(x) = p_{12}u(c-1) + p_1 u(c-x_1) + p_2 u(c+x_1-1) + p_\emptyset u(c)$$

First-Order Condition

$$Eu(x) = p_{12}u(c-1) + p_1 u(c-x_1) + p_2 u(c+x_1-1) + p_{\emptyset} u(c)$$

$$\frac{\partial Eu(x)}{\partial x_1} = -p_1 u'(c-x_1) + p_2 u'(c+x_1-1)$$

Taylor series of first order around $c - 0.5$

$$\frac{\partial Eu(x)}{\partial x_1} \approx -p_1[u'(c-0.5) + (0.5-x_1)u''(c-0.5)] + p_2[u'(c-0.5) + (x_1-0.5)u''(c-0.5)]$$

Solution

$$0 = -p_1[u'(c-0.5) + (0.5 - x_1)u''(c-0.5)] \\ + p_2[u'(c-0.5) + (x_1 - 0.5)u''(c-0.5)]$$

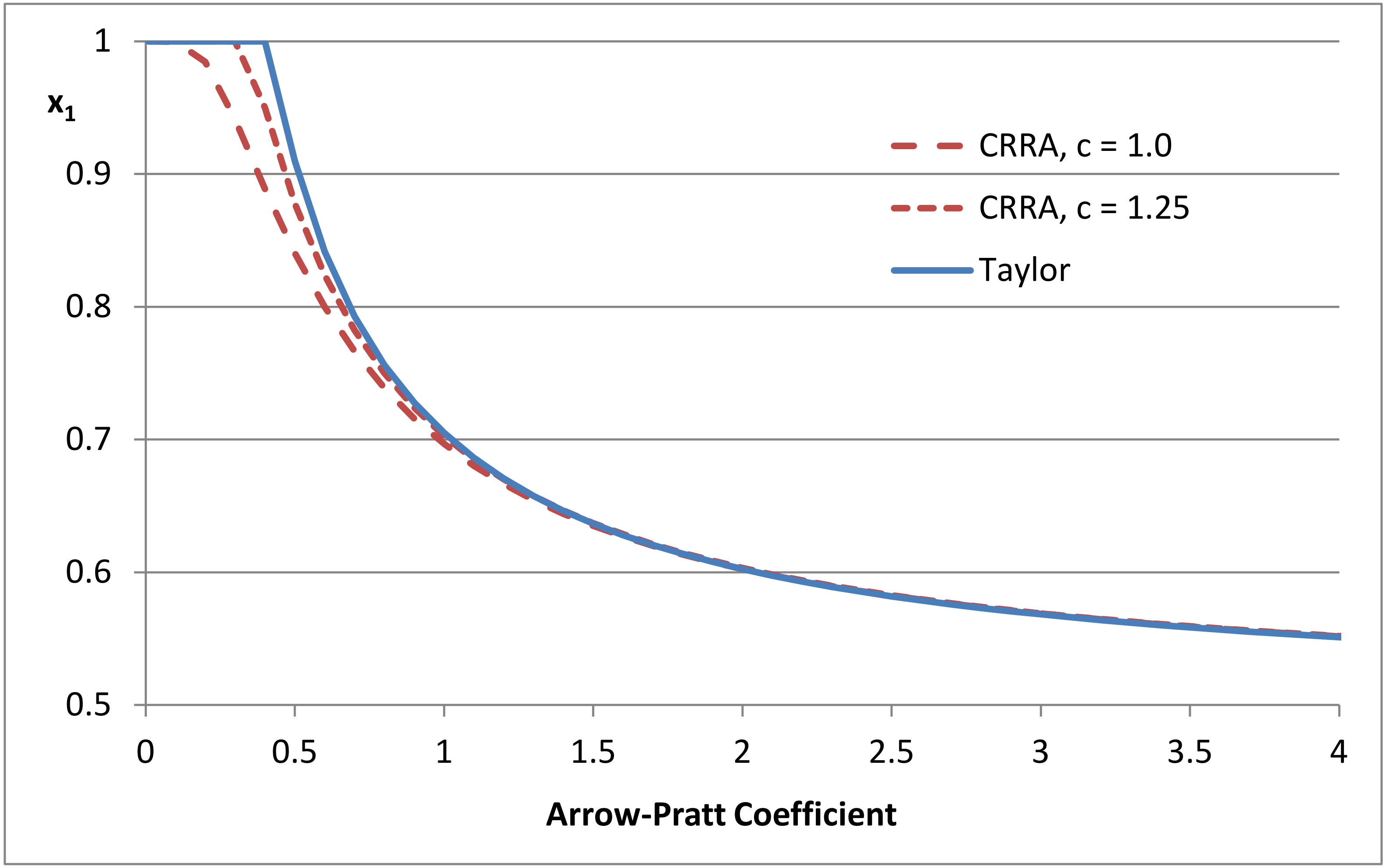
$$\Leftrightarrow 0 = -p_1[1 + (0.5 - x_1)\eta] + p_2[1 + (x_1 - 0.5)\eta]$$

with the Arrow-Pratt coefficient of risk aversion $\eta = -\frac{u''(c-0.5)}{u'(c-0.5)}$

Solution:

$$x_1 = \frac{p_2 - p_1 + 0.5\eta(p_1 + p_2)}{\eta(p_1 + p_2)}$$

Example


1
0.9
0.8
0.7
0.6
0.5
x_1
CRRA, c = 1.0
CRRA, c = 1.25
Taylor
0
0.5
1
1.5
2
2.5
3
3.5
4
Arrow-Pratt Coefficient

Multivariate Case

Langrangian Approach

$$\lambda + \sum_{i=1}^{n} x_i \left(\sum_{I:i,j \in I} p_I \right) = - \sum_{I:j \in I} p_I (1 - 0.5\eta) , \qquad j = 1, \dots, n$$

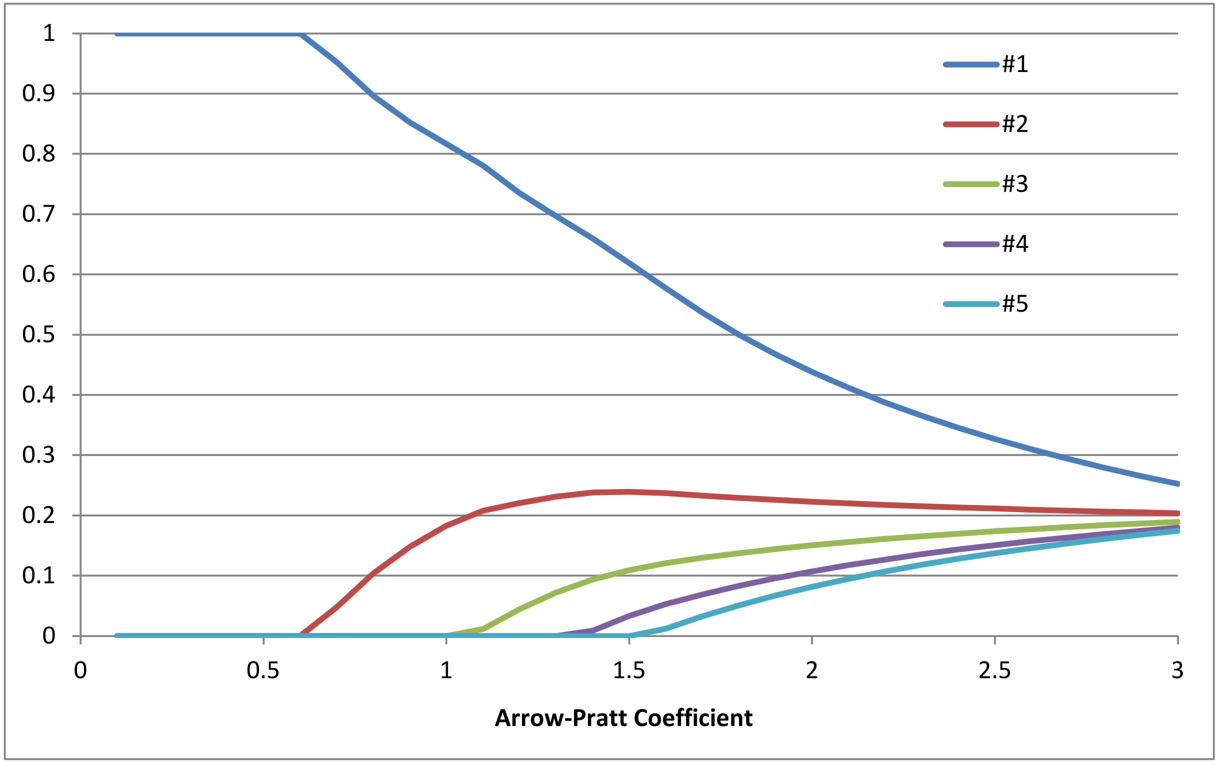
$$I \subset \{1, \dots, n\}$$

Linear system with $n + 1$ equations and $n + 1$ variables

but $\mathcal{O}(2^n)$

Solutions for n = 5

$p_j = j\,\%;\ \rho = 0.3$



Round-up

Expected-Utility approach yields reasonable solutions

- in line with economic intuition
- easy to calculate with analytical tractability
- operationalizable via Arrow-Pratt coefficient of risk aversion

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Summary

- Allocation problem needs a precise formulation
  - What is the target function?

- Minimization of spectral risk measures may lead to economically implausible solutions
  - Tendency to corner solutions
    because of translation invariance and positive homogeneity

- Application of expected utility theory is advantageous
  - Natural target function
  - Measuring risk via Arrow-Pratt coefficient of risk aversion

Implications for Risk Governance

- Awareness for the scope of models and measures
  - Know what your models are doing!
  - Know what they are good for!
  - Know what they are not good for!
- Interdisciplinary teams for risk model design
- (Basic) Understanding of models and measures at C-level (Chief Risk Officer)

„The moment you have worked out an answer, start checking it – it probably isn't right."

Edmund C. Berkeley